

# Construction and classification of holomorphic vertex operator algebras

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We develop an orbifold theory for finite, cyclic groups acting on holomorphic vertex operator algebras. Then we show that Schellekens' classification of  $V_1$ -structures of meromorphic conformal field theories of central charge 24 is a theorem on vertex operator algebras. Finally we use these results to construct some new holomorphic vertex operator algebras of central charge 24 as lattice orbifolds.

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## 1 Introduction

Vertex algebras have been introduced into mathematics by R. E. Borcherds [B1]. These algebras together with their representations give a mathematically rigorous description of conformal field theories. A prominent application is Borcherds' proof of the moonshine conjecture for the monster [B2]. Two important problems in the theory of vertex algebras are the classification of vertex operator algebras and the construction of new vertex operator algebras as orbifolds.

In this paper we prove the following results.

Let  $V$  be a simple, rational,  $C_2$ -cofinite, self-contragredient vertex operator algebra of CFT-type. Assume that all irreducible  $V$ -modules are simple currents. Then the fusion algebra  $\mathcal{V}(V)$  of  $V$  is the group algebra  $\mathbb{C}[D]$  of some finite abelian group  $D$  (Theorem 3.1). Suppose in addition that the irreducible modules different from  $V$  all have positive conformal weights. Then the modulo 1 reduction  $q_\Delta$  of the conformal weights provides  $D$  with the structure of a discriminant form and Zhu's representation is up to a character the Weil representation of  $D$  (Theorem 3.4). The direct sum of the irreducible  $V$ -modules has the structure of an abelian intertwining algebra whose associated quadratic form is  $-q_\Delta$  (Theorem 4.1). Restricting the sum to an isotropic subgroup gives a vertex operator algebra extending  $V$  (Theorem 4.2).

Now we assume in addition that  $V$  is holomorphic and we let  $G = \langle g \rangle$  be a finite, cyclic group of automorphisms of  $V$  of order  $n$ . Then the fixed-point subalgebra  $V^G$  has group-like fusion and the fusion group  $D$  is a central extension of  $\mathbb{Z}_n$  by  $\mathbb{Z}_n$  whose isomorphism type is determined by the conformal weight of the twisted module  $V(g)$  (Theorem 5.12). Let  $N$  be the level of  $D$  with respect to  $q_\Delta$ . Then the twisted traces of  $V$  are modular forms for  $\Gamma(N)$  if 24 divides  $c$  and for  $\Gamma(\text{lcm}(N, 3))$  otherwise (Theorem 5.14). Suppose the  $g^j$ -twisted modules of  $V$  have positive conformal weights for  $j \neq 0$ . Then the direct sum of all irreducible  $V^G$ -modules is an abelian intertwining algebra and the restriction of the sum to an isotropic subgroup  $H$  of  $D$  is a vertex operator algebra extending  $V^G$  called the orbifold of  $V$  with respect to  $G$  and  $H$  (Theorem 5.15).

Next we consider the classification problem. Let  $V$  be a simple, rational,  $C_2$ -cofinite, holomorphic vertex operator algebra of CFT-type and central charge 24. Then either  $V_1 = 0$ ,  $\dim(V_1) = 24$  and  $V$  is isomorphic to the Leech lattice vertex operator algebra or  $V_1$  is one of the 69 semisimple Lie algebras described in Table 1 of [S] (Theorem 6.4).

As an application of our results we construct new holomorphic vertex operator algebras of central charge 24 with  $V_1$  given by  $A_{2,1}B_{2,1}E_{6,4}$ ,  $A_{4,5}^2$ ,  $A_{2,6}D_{4,12}$ ,  $A_{1,1}C_{5,3}G_{2,2}$  and  $C_{4,10}$  as lattice orbifolds (Theorem 8.1).

Another application is Carnahan's proof of Norton's generalised moonshine conjecture [C].

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After completion of this work we learned of [DRX] which has some overlap with our Section 5, in particular Proposition 5.6.

## 2 Modular invariance and the Verlinde formula

In this section we recall some of the main results that we will use.

Let  $V$  be a rational,  $C_2$ -cofinite vertex operator algebra of central charge  $c$  and  $W$  an irreducible  $V$ -module of conformal weight  $\rho(W)$ . Then for  $v \in V$  we define the formal sum

$$T_W(v, q) = \text{tr}_W o(v) q^{L_0 - c/24} = q^{\rho(W) - c/24} \sum_{n=0}^{\infty} \text{tr}_{W_{\rho+n}} o(v) q^n$$

where  $o(v) = v_{\text{wt}(v)-1}$  for homogeneous  $v$ , extended linearly to  $V$ . Zhu [Z] has shown

### Theorem 2.1

Let  $V$  be a rational,  $C_2$ -cofinite vertex operator algebra of central charge  $c$  and  $W \in \text{Irr}(V)$ , the set of isomorphism classes of irreducible  $V$ -modules. Then

- i) Let  $q = e^{2\pi i\tau}$ . Then the formal sum  $T_W(v, \tau)$  converges to a holomorphic function on the complex upper halfplane  $H$ .
- ii) Let  $v \in V_{[k]}$  be of weight  $k$  with respect to Zhu's second grading. Then there is a representation

$$\rho_V : \text{SL}_2(\mathbb{Z}) \rightarrow \text{GL}(\mathcal{V}(V))$$

of  $\text{SL}_2(\mathbb{Z})$  on the fusion algebra  $\mathcal{V}(V) = \bigoplus_{W \in \text{Irr}(V)} \mathbb{C}W$  of  $V$  such that

$$T_W(v, \gamma\tau) = (c\tau + d)^k \sum_{M \in \text{Irr}(V)} \rho_V(\gamma)_{W,M} T_M(v, \tau)$$

for all  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ .

Under the conditions of the theorem the central charge of  $V$  and the conformal weights of the irreducible modules of  $V$  are rational [DLM].

We denote the images of the standard generators  $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  and  $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  of  $\text{SL}_2(\mathbb{Z})$  under Zhu's representation  $\rho_V$  by  $\mathcal{S}$  and  $\mathcal{T}$ . Then

$$\mathcal{T}_{M,N} = \delta_{M,N} e((\rho(M) - c/24))$$

where  $e(x)$  denotes  $e^{2\pi i x}$ . The  $\mathcal{S}$ -matrix is related to the fusion coefficients by the Verlinde formula proved for vertex operator algebras by Huang [H3].

### Theorem 2.2

Let  $V$  be a simple, rational,  $C_2$ -cofinite, self-contragredient vertex operator algebra of CFT-type. Then

- i) The fusion coefficients are given by

$$\mathcal{N}_{M,N}^W = \sum_{U \in \text{Irr}(V)} \frac{\mathcal{S}_{M,U} \mathcal{S}_{N,U} \mathcal{S}_{W',U}}{\mathcal{S}_{V,U}}.$$

- ii) The matrix  $\mathcal{S}$  is symmetric and  $\mathcal{S}^2$  is the permutation matrix sending  $M$  to its contragredient module  $M'$ .

Now let  $V$  be a simple, rational,  $C_2$ -cofinite vertex operator algebra of central charge  $c$  which has only one irreducible module, i.e.  $V$  is holomorphic and let  $G = \langle g \rangle$  be a finite, cyclic group of automorphisms of  $V$  of order  $n$ . Then for each  $h \in G$  there is a unique irreducible  $h$ -twisted  $V$ -module  $V(h)$  of conformal weight  $\rho(V(h)) \in \mathbb{Q}$  and a representation

$$\phi_h : G \rightarrow \text{Aut}_{\mathbb{C}}(V(h))$$

of  $G$  on the vector space  $V(h)$  such that

$$\phi_h(k)Y_{V(h)}(v, z)\phi_h(k)^{-1} = Y_{V(h)}(kv, z)$$

for all  $k \in G$  and  $v \in V$  [DLM]. By Schur's lemma this representation is unique up to multiplication by an  $n$ -th root of unity. Setting  $h = g^i$ ,  $k = g^j$  we define the twisted trace functions

$$T(v, i, j, q) = \text{tr}_{V(g^i)} o(v) \phi_{g^i}(g^j) q^{L_0 - c/24}.$$

Dong, Li and Mason [DLM] have shown

**Theorem 2.3**

Let  $V$  be a simple, rational,  $C_2$ -cofinite, holomorphic vertex operator algebra of central charge  $c$  and  $G = \langle g \rangle$  a finite, cyclic group of automorphisms of  $V$  of order  $n$ . Then

- i) Let  $q = e^{2\pi i \tau}$ . Then the twisted trace function  $T(v, i, j, \tau)$  converges to a holomorphic function on  $H$ .
- ii) For homogeneous  $v \in V_{[k]}$  the twisted trace functions satisfy

$$T(v, i, j, \gamma\tau) = \sigma(i, j, \gamma)(c\tau + d)^k T(v, (i, j)\gamma, \tau)$$

for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ . The constants  $\sigma(i, j, \gamma) \in \mathbb{C}$  depend only on  $i, j, \gamma$  and the choice of the  $\phi_i$ .

### 3 Simple currents

In this section we show that a rational vertex operator algebra satisfying certain regularity conditions and whose modules are all simple currents has group-like fusion and that the conformal weights define a quadratic form on the fusion group. We also show that Zhu's representation is up to a character the Weil representation in this case.

Let  $V$  be a rational,  $C_2$ -cofinite vertex operator algebra of CFT-type. It is well-known that the fusion algebra  $\mathcal{V}(V)$  of  $V$  is a finite-dimensional, associative, commutative algebra over  $\mathbb{C}$  with unit  $V$ . Let  $\text{Irr}(V) = \{W^\alpha \mid \alpha \in D\}$  be the

set of isomorphism classes of irreducible  $V$ -modules. We assume that all  $W^\alpha$  are simple currents. Then we can define a composition on  $D$  by

$$W^\alpha \boxtimes_V W^\beta \cong W^{\alpha+\beta}.$$

If  $V$  is in addition simple and self-contragredient then  $D$  actually is an abelian group with identity  $W^0 = V$ .

**Theorem 3.1**

*Let  $V$  be a simple, rational,  $C_2$ -cofinite, self-contragredient vertex operator algebra of CFT-type. Assume that all irreducible  $V$ -modules are simple currents. Then the fusion algebra  $\mathcal{V}(V)$  of  $V$  is the group algebra  $\mathbb{C}[D]$  of some finite abelian group  $D$ , i.e.*

$$W^\alpha \boxtimes_V W^\beta \cong W^{\alpha+\beta}$$

*for all  $\alpha, \beta \in D$ . The neutral element is given by  $W^0 = V$  and the inverse of  $\alpha$  by  $W^{-\alpha} = W^{\alpha'} \cong (W^\alpha)'$ .*

*Proof:* We only have to prove the last statement. By the Verlinde formula we have

$$\delta_{\alpha\beta} = \mathcal{N}_{\alpha 0}^\beta = \sum_{\gamma \in D} \mathcal{S}_{\alpha\gamma} \mathcal{S}_{\gamma\beta'}.$$

The symmetry of the  $\mathcal{S}$ -matrix implies

$$\mathcal{N}_{\alpha\alpha'}^0 = \sum_{\gamma \in D} \frac{\mathcal{S}_{\alpha\gamma} \mathcal{S}_{\alpha'\gamma} \mathcal{S}_{0\gamma}}{\mathcal{S}_{0\gamma}} = \sum_{\gamma \in D} \mathcal{S}_{\alpha\gamma} \mathcal{S}_{\gamma\alpha'} = \delta_{\alpha\alpha} = 1.$$

This proves the theorem.  $\square$

If  $V$  is as in the theorem we say that  $V$  has group-like fusion. We will assume this from now on. We denote the conformal weight of  $W^\alpha$  by  $\rho(W^\alpha)$  and define a map

$$q_\Delta : D \rightarrow \mathbb{Q}/\mathbb{Z}$$

by

$$\alpha \mapsto \rho(W^\alpha) \pmod{1}.$$

We also define

$$b_\Delta : D \times D \rightarrow \mathbb{Q}/\mathbb{Z}$$

by

$$b_\Delta(\alpha, \beta) = q_\Delta(\alpha + \beta) - q_\Delta(\alpha) - q_\Delta(\beta) \pmod{1}.$$

**Proposition 3.2**

*Suppose  $V$  has group-like fusion with fusion group  $D$  and the modules  $W^\alpha$  have positive conformal weights for  $\alpha \neq 0$ . Then*

$$\mathcal{S}_{00} = \mathcal{S}_{0\alpha} = \frac{1}{\sqrt{|D|}}$$

*for all  $\alpha \in D$ .*

*Proof:* We have  $\mathcal{S}_{00} = \mathcal{S}_{0\alpha} = \mathcal{S}_{\alpha 0} \in \mathbb{R}^+ [\text{DJX}]$  so that

$$1 = \delta_{00} = (\mathcal{S}^2)_{00} = \sum_{\gamma \in D} \mathcal{S}_{\gamma 0} \mathcal{S}_{0\gamma} = |D| \mathcal{S}_{00}^2.$$

This proves the statement.  $\square$

**Proposition 3.3**

Suppose  $V$  has group-like fusion with fusion group  $D$  and the modules  $W^\alpha$  have positive conformal weights for  $\alpha \neq 0$ . Then

$$\mathcal{S}_{\alpha\beta} = \mathcal{S}_{00} e(-b_\Delta(\alpha, \beta))$$

for all  $\alpha, \beta \in D$ .

*Proof:* The relation  $TSTST = S$  in  $\text{SL}_2(\mathbb{Z})$  implies

$$\mathcal{S}_{\alpha\beta} = \sum_{\gamma \in D} \mathcal{S}_{\alpha\gamma} \mathcal{S}_{\gamma\beta} e(q_\Delta(\alpha) + q_\Delta(\beta) + q_\Delta(\gamma) - c/8).$$

By the Verlinde formula we have

$$\begin{aligned} \mathcal{S}_{\alpha+\beta, \gamma} &= \sum_{\delta \in D} \mathcal{S}_{\delta\gamma} \delta_{\alpha+\beta, \delta} \\ &= \sum_{\delta \in D} \mathcal{S}_{\delta\gamma} \mathcal{N}_{\alpha\beta}^\delta \\ &= \sum_{\delta, \rho \in D} \mathcal{S}_{\delta\gamma} \frac{\mathcal{S}_{\alpha\rho} \mathcal{S}_{\beta\rho} \mathcal{S}_{-\delta, \rho}}{\mathcal{S}_{0\rho}} \\ &= \sum_{\rho \in D} \frac{\mathcal{S}_{\alpha\rho} \mathcal{S}_{\beta\rho}}{\mathcal{S}_{0\rho}} \sum_{\delta \in D} \mathcal{S}_{\delta\gamma} \mathcal{S}_{-\delta, \rho} \\ &= \sum_{\rho \in D} \frac{\mathcal{S}_{\alpha\rho} \mathcal{S}_{\beta\rho}}{\mathcal{S}_{0\rho}} \delta_{\gamma\rho} \\ &= \frac{\mathcal{S}_{\alpha\gamma} \mathcal{S}_{\beta\gamma}}{\mathcal{S}_{0\gamma}} \end{aligned}$$

because

$$\sum_{\delta \in D} \mathcal{S}_{\delta\gamma} \mathcal{S}_{-\delta, \rho} = \sum_{\delta, \mu \in D} \mathcal{S}_{\gamma\delta} \delta_{-\delta, \mu} \mathcal{S}_{\mu\rho} = \sum_{\delta, \mu \in D} \mathcal{S}_{\gamma\delta} (\mathcal{S}^2)_{\delta\mu} \mathcal{S}_{\mu\rho} = (\mathcal{S}^4)_{\gamma\rho} = \delta_{\gamma\rho}.$$

Hence

$$\mathcal{S}_{\alpha+\beta, \gamma} \mathcal{S}_{0\gamma} = \mathcal{S}_{\alpha\gamma} \mathcal{S}_{\beta\gamma}$$

so that

$$\begin{aligned}
\mathcal{S}_{\alpha\beta} &= \sum_{\gamma \in D} \mathcal{S}_{\alpha\gamma} \mathcal{S}_{\gamma\beta} e(q_{\Delta}(\alpha) + q_{\Delta}(\beta) + q_{\Delta}(\gamma) - c/8) \\
&= \sum_{\gamma \in D} \mathcal{S}_{\alpha+\beta, \gamma} \mathcal{S}_{0\gamma} e(q_{\Delta}(\alpha) + q_{\Delta}(\beta) + q_{\Delta}(\gamma) - c/8) \\
&= e(q_{\Delta}(\alpha) + q_{\Delta}(\beta) - c/8) \sum_{\gamma \in D} \mathcal{S}_{\alpha+\beta, \gamma} \mathcal{S}_{0\gamma} e(q_{\Delta}(\gamma)) \\
&= e(q_{\Delta}(\alpha) + q_{\Delta}(\beta) - c/8) \mathcal{S}_{\alpha+\beta, 0} e(-q_{\Delta}(\alpha + \beta) + c/8) \\
&= e(-b_{\Delta}(\alpha, \beta)) \mathcal{S}_{\alpha+\beta, 0}
\end{aligned}$$

since

$$\begin{aligned}
\mathcal{S}_{\alpha 0} &= \sum_{\gamma \in D} \mathcal{S}_{\alpha\gamma} \mathcal{S}_{\gamma 0} e(q_{\Delta}(\alpha) + q_{\Delta}(\gamma) - c/8) \\
&= e(q_{\Delta}(\alpha) - c/8) \sum_{\gamma \in D} \mathcal{S}_{\alpha\gamma} \mathcal{S}_{\gamma 0} e(q_{\Delta}(\gamma)).
\end{aligned}$$

This finishes the proof.  $\square$

### Theorem 3.4

Let  $V$  be a simple, rational,  $C_2$ -cofinite, self-contragredient vertex operator algebra of CFT-type. Suppose  $V$  has group-like fusion with fusion group  $D$  and the modules  $W^{\alpha}$  have positive conformal weights for  $\alpha \neq 0$ . Then

$$\begin{aligned}
\mathcal{S}_{\alpha\beta} &= \frac{1}{\sqrt{|D|}} e(-b_{\Delta}(\alpha, \beta)), \\
\mathcal{T}_{\alpha\beta} &= e(q_{\Delta}(\alpha) - c/24) \delta_{\alpha\beta}.
\end{aligned}$$

Moreover  $q_{\Delta}$  is a quadratic form on  $D$  and  $b_{\Delta}$  the associated bilinear form.

*Proof:* The formula for  $\mathcal{T}$  is clear. The formula for  $\mathcal{S}$  follows from the previous two results. The relation

$$\mathcal{S}_{\alpha\gamma} \mathcal{S}_{\beta\gamma} = \mathcal{S}_{\alpha+\beta, \gamma} \mathcal{S}_{0\gamma}$$

together with Proposition 3.3 shows that  $b_{\Delta}$  is bilinear. We have  $q_{\Delta}(0) = 0 \pmod{1}$  and  $q_{\Delta}(\alpha) = q_{\Delta}(-\alpha)$  for all  $\alpha \in D$ . This implies that  $q_{\Delta}$  is a quadratic form with associated bilinear form  $b_{\Delta}$ .  $\square$

### Proposition 3.5

Suppose  $V$  has group-like fusion with fusion-group  $D$  and the modules  $W^{\alpha}$  have positive conformal weights for  $\alpha \neq 0$ . Then the bilinear form  $b_{\Delta}$  is non-degenerate.

*Proof:* We have

$$\delta_{\alpha, -\beta} = (\mathcal{S}^2)_{\alpha\beta} = \sum_{\gamma \in D} \mathcal{S}_{\alpha\gamma} \mathcal{S}_{\beta\gamma} = \frac{1}{|D|} \sum_{\gamma \in D} e(-b_{\Delta}(\alpha + \beta, \gamma))$$

so that

$$\delta_{\alpha 0} = \frac{1}{|D|} \sum_{\gamma \in D} e(-b_{\Delta}(\alpha, \gamma)).$$

This implies the statement.  $\square$

It follows that  $(D, q_{\Delta})$  is a discriminant form. The group algebra  $\mathbb{C}[D]$  carries two representations of the metaplectic group  $\text{Mp}_2(\mathbb{Z})$ , one coming from Zhu's theorem and the other being the Weil representation [B3]. These representations commute and the explicit formulas show that they are related by a character. As a consequence we have

**Theorem 3.6**

*Let  $V$  be a simple, rational,  $C_2$ -cofinite, self-contragredient vertex operator algebra of CFT-type. Suppose  $V$  has group-like fusion with fusion group  $D$  and the modules  $W^{\alpha}$  have positive conformal weights for  $\alpha \neq 0$ . Then the central charge  $c$  of  $V$  is a integer and  $D$  is a discriminant form under  $q_{\Delta}$  of signature*

$$\text{sign}(D) = c \pmod{8}.$$

## 4 Abelian intertwining algebras

In this section we show that the irreducible modules of a rational vertex operator algebra  $V$  satisfying certain regularity conditions and with group-like fusion form an abelian intertwining algebra whose quadratic form is determined by the conformal weights. Restricting the sum to an isotropic subgroup gives a vertex operator algebra extending  $V$ .

Let  $D$  be a finite abelian group and  $(F, \Omega)$  a normalised abelian 3-cocycle on  $D$  with coefficients in  $\mathbb{C}^*$ , i.e. the maps

$$\begin{aligned} F : D \times D \times D &\rightarrow \mathbb{C}^* \\ \Omega : D \times D &\rightarrow \mathbb{C}^* \end{aligned}$$

satisfy

$$\begin{aligned} F(\alpha, \beta, \gamma) F(\alpha, \beta, \gamma + \delta)^{-1} F(\alpha, \beta + \gamma, \delta) F(\alpha + \beta, \gamma, \delta)^{-1} F(\beta, \gamma, \delta) &= 1 \\ F(\alpha, \beta, \gamma)^{-1} \Omega(\alpha, \beta + \gamma) F(\beta, \gamma, \alpha)^{-1} &= \Omega(\alpha, \beta) F(\beta, \alpha, \gamma)^{-1} \Omega(\alpha, \gamma) \\ F(\alpha, \beta, \gamma) \Omega(\alpha + \beta, \gamma) F(\gamma, \alpha, \beta) &= \Omega(\beta, \gamma) F(\alpha, \gamma, \beta) \Omega(\alpha, \gamma) \end{aligned}$$

and

$$\begin{aligned} F(\alpha, \beta, 0) &= F(\alpha, 0, \gamma) = F(0, \beta, \gamma) = 1 \\ \Omega(\alpha, 0) &= \Omega(0, \beta) = 1 \end{aligned}$$



for all  $\alpha, \beta, \gamma, \delta \in D$ . Define

$$B : D \times D \times D \rightarrow \mathbb{C}^*$$

by

$$B(\alpha, \beta, \gamma) = F(\beta, \alpha, \gamma)^{-1} \Omega(\alpha, \beta) F(\alpha, \beta, \gamma).$$

We also define a quadratic form

$$q_\Omega : D \rightarrow \mathbb{Q}/\mathbb{Z}$$

by

$$\Omega(\alpha, \alpha) = e(q_\Omega(\alpha))$$

for all  $\alpha \in D$ . We denote the corresponding bilinear form by  $b_\Omega$ . The level of  $(F, \Omega)$  is the smallest positive integer  $N$  such that  $Nq_\Omega(\alpha) = 0 \pmod{1}$  for all  $\alpha \in D$ .

An abelian intertwining algebra of level  $N$  associated to  $D, F$  and  $\Omega$  is a  $\mathbb{C}$ -vector space  $V$  with a  $\frac{1}{N}\mathbb{Z}$ -grading and a  $D$ -grading

$$V = \bigoplus_{n \in \frac{1}{N}\mathbb{Z}} V_n = \bigoplus_{\alpha \in D} V^\alpha$$

such that

$$V^\alpha = \bigoplus_{n \in \frac{1}{N}\mathbb{Z}} V_n^\alpha$$

where  $V_n^\alpha = V_n \cap V^\alpha$  equipped with a state-field correspondence

$$\begin{aligned} Y : V &\rightarrow \text{End}(V)[[z^{-1/N}, z^{1/N}]] \\ a &\mapsto Y(a, z) = \sum_{n \in \frac{1}{N}\mathbb{Z}} a_n z^{-n-1} \end{aligned}$$

and with two distinguished vectors  $\mathbf{1} \in V_0^0$ ,  $\omega \in V_2^0$  satisfying the following conditions. For  $\alpha, \beta \in D$ ,  $a, b \in V$  and  $n \in \frac{1}{N}\mathbb{Z}$

$$\begin{aligned} a_n V^\beta &\subset V^{\alpha+\beta} \quad \text{if } a \in V^\alpha \\ a_n b &= 0 \quad \text{for } n \text{ sufficiently large} \\ Y(a, z)\mathbf{1} &\in V[[z]] \quad \text{and} \quad \lim_{z \rightarrow 0} Y(a, z)\mathbf{1} = a \\ Y(a, z)|_{V^\beta} &= \sum_{n \in b_\Omega(\alpha, \beta) + \mathbb{Z}} a_n z^{-n-1} \quad \text{if } a \in V^\alpha, \end{aligned}$$

the Jacobi identity

$$\begin{aligned} &x^{-1} \left( \frac{y-z}{x} \right)^{b_\Omega(\alpha, \beta)} \delta \left( \frac{y-z}{x} \right) Y(a, y) Y(b, z) c \\ &\quad - B(\alpha, \beta, \gamma) x^{-1} \left( \frac{z-y}{e^{\pi i} x} \right)^{b_\Omega(\alpha, \beta)} \delta \left( \frac{z-y}{-x} \right) Y(b, z) Y(a, y) c \\ &= F(\alpha, \beta, \gamma) z^{-1} \delta \left( \frac{y-x}{z} \right) Y(Y(a, x)b, z) \left( \frac{y-x}{z} \right)^{-b_\Omega(\alpha, \gamma)} c \end{aligned}$$

holds for all  $a \in V^\alpha$ ,  $b \in V^\beta$  and  $c \in V^\gamma$ , the operators  $L_n$  defined by

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$$

satisfy

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{m^3 - m}{12} \delta_{m+n} c$$

for some  $c \in \mathbb{C}$  and

$$\begin{aligned} L_0 a &= n a \quad \text{for } a \in V_n, \\ \frac{d}{dz} Y(a, z) &= Y(L_{-1} a, z). \end{aligned}$$

Note that the cohomology class of  $(F, \Omega)$  in  $H_{\text{ab}}^3(G, \mathbb{C}^*)$  is determined by  $q_\Omega$ . We also remark that rescaling the intertwining operators amounts to changing  $(F, \Omega)$  by a coboundary ([DL1], Remarks 12.22, 12.23).

A consequence of the Jacobi identity is the skew-symmetry formula

$$Y(a, z)b = \frac{1}{\Omega(\beta, \alpha)} e^{zL_{-1}} Y(b, e^{-\pi i} z) a$$

for  $a \in V^\alpha$ ,  $b \in V^\beta$ .

It is well-known that the vertex algebra associated with an even lattice together with its irreducible modules forms an abelian intertwining algebra where  $q_\Omega$  is determined by the conformal weights ([DL1], Theorem 12.24). More generally we have

#### Theorem 4.1

*Let  $V$  be a simple, rational,  $C_2$ -cofinite, self-contragredient vertex operator algebra of CFT-type. Assume that  $V$  has group-like fusion, i.e. the fusion algebra is  $\mathbb{C}[D]$  for some finite abelian group  $D$ , and that the irreducible  $V$ -modules  $W^\alpha$ ,  $\alpha \neq 0$  have positive conformal weights. Then*

$$W = \bigoplus_{\alpha \in D} W^\alpha$$

*can be given the structure of an abelian intertwining algebra with normalised abelian 3-cocycle  $(F, \Omega)$  such that*

$$q_\Omega = -q_\Delta.$$

*Proof:* Choosing non-trivial intertwining operators between the modules  $W^\alpha$  of  $V$  the Jacobi identity defines maps  $F$  and  $\Omega$ . Huang has shown that  $(F, \Omega)$  is an abelian 3-cocycle on  $D$  (cf. [H2], Theorem 3.7 and [H1]). Any abelian 3-cocycle is cohomologous to a normalised one so that  $W$  is an abelian intertwining algebra. Furthermore the modules of  $V$  form a modular tensor category with the twist morphisms  $\theta_\alpha : W^\alpha \rightarrow W^\alpha$  on the irreducible modules given by

$\theta_\alpha = e(q_\Delta(\alpha)) \text{id}_{W^\alpha}$  (cf. [H4]). Since  $V$  has group-like fusion the braiding isomorphism

$$c_{\alpha,\beta} : W^\alpha \boxtimes W^\beta \rightarrow W^\beta \boxtimes W^\alpha$$

is for  $\alpha = \beta$  given by

$$c_{\alpha,\alpha} = e(q_\Omega(\alpha))e(2q_\Delta(\alpha)) \text{id}_{W^\alpha \boxtimes W^\alpha}.$$

We have

$$\text{tr}_{W^\alpha} \theta_\alpha = \text{tr}_{W^\alpha \boxtimes W^\alpha} c_{\alpha,\alpha}$$

(cf. Proposition 2.32 in [DGNO]) so that

$$e(q_\Delta(\alpha)) \text{tr}_{W^\alpha} \text{id}_{W^\alpha} = e(q_\Omega(\alpha))e(2q_\Delta(\alpha)) \text{tr}_{W^{2\alpha}} \text{id}_{W^{2\alpha}}.$$

The trace

$$d_\alpha = \text{tr}_{W^\alpha} \text{id}_{W^\alpha}$$

is the categorial quantum dimension of  $W^\alpha$ . For Huang's construction under the positivity assumption on the conformal weights it coincides with the definition of the quantum dimension as the limit of a certain character ratio and  $d_\alpha = 1$  if  $W^\alpha$  is a simple current (cf. [DJX], [DLN]). Then the above identity implies

$$e(q_\Omega(\alpha)) = e(-q_\Delta(\alpha))$$

for all  $\alpha \in D$ . This proves the theorem.  $\square$

This result is also stated as Theorem 2.7 in [HS]. In the above proof we fill a gap pointed out by S. Carnahan. The theorem can also be derived from Bantay's formula for the Frobenius-Schur indicator and the fact that the theorem holds for abelian intertwining algebras associated with even lattices.

As an application we obtain

#### Theorem 4.2

*Let  $V$  be a simple, rational,  $C_2$ -cofinite, self-contragredient vertex operator algebra of CFT-type with group-like fusion and fusion group  $D$ . Suppose that the irreducible  $V$ -modules  $W^\alpha$ ,  $\alpha \in D \setminus \{0\}$  have positive conformal weights. Let  $H$  be an isotropic subgroup of  $D$  with respect to  $q_\Delta$ . Then*

$$W^H = \bigoplus_{\gamma \in H} W^\gamma$$

*admits the structure of a simple, rational,  $C_2$ -cofinite, self-contragredient vertex operator algebra of CFT-type extending the vertex operator algebra structure on  $V$ . If  $H = H^\perp$  then  $W^H$  is holomorphic.*

*Proof:* Suppose  $q_\Delta|_H = q_\Omega|_H = 0 \pmod{1}$ . Then  $(F|_H, \Omega|_H)$  is cohomologous to the trivial 3-cocycle in  $H_{\text{ab}}^3(H, \mathbb{C}^*)$ . Hence the abelian intertwining algebra  $W^H$  admits the structure of a vertex operator algebra upon rescaling of the intertwining operators. The irreducible modules of  $W^H$  are given by

$$W^{H,\gamma} = \bigoplus_{\alpha \in \gamma + H} W^\alpha$$

where  $\gamma$  ranges over  $H^\perp/H$  (cf. [Y]).  $\square$

## 5 Orbifolds

Let  $G = \langle g \rangle$  be a finite, cyclic group of order  $n$  acting on a holomorphic vertex operator algebra  $V$ . We show that the fixed-point subalgebra  $V^G$  has group-like fusion and that the fusion group is a central extension of  $\mathbb{Z}_n$  by  $\mathbb{Z}_n$  whose isomorphism type is fixed by the conformal weight of the twisted module  $V(g)$ . We also determine the  $\mathcal{S}$ -matrix and describe the level of the trace functions. If the twisted modules  $V(g^j)$  have positive conformal weights for  $j \neq 0$  then the direct sum of the irreducible  $V^G$ -modules is an abelian intertwining algebra and the restriction of the sum to an isotropic subgroup with respect to the conformal weights is a vertex operator algebra extending  $V^G$ . Our approach is inspired by Miyamoto's theory of  $\mathbb{Z}_3$ -orbifolds [M1].

Let  $V$  be a simple, rational,  $C_2$ -cofinite, holomorphic vertex operator algebra of CFT-type and  $G = \langle g \rangle$  a finite, cyclic group of automorphisms of  $V$  of order  $n$ . Then we have (cf. [CM], [M2])

### Theorem 5.1

*The fixed-point subalgebra  $V^G$  is a simple, rational,  $C_2$ -cofinite, self-contra-  
gradient vertex operator algebra of CFT-type. Every irreducible  $V^G$ -module is  
isomorphic to a  $V^G$ -submodule of the irreducible  $g^i$ -twisted  $V$ -module  $V(g^i)$  for  
some  $i$ .*

Recall that for each  $h \in G$  there is a representation  $\phi_h : G \rightarrow \text{Aut}_{\mathbb{C}}(V(h))$  of  $G$  on the vector space  $V(h)$  such that  $\phi_h(k)Y_{V(h)}(v, z)\phi_h(k)^{-1} = Y_{V(h)}(kv, z)$  for all  $k \in G$  and  $v \in V$  and these representations are unique up to multiplication by an  $n$ -th root of unity. If  $h$  is the identity we can and will assume that  $\phi_h(k) = k$  for all  $k \in G$ . We will often write  $\phi_i$  for  $\phi_{g^i}$ . We denote the eigenspace of  $\phi_j(g)$  in  $V(g^j)$  with eigenvalue  $e(l/n) = e^{2\pi i l/n}$  by  $W^{(j,l)}$ , i.e.

$$W^{(j,l)} = \{w \in V(g^j) \mid \phi_j(g)v = e(l/n)v\}.$$

The twisted trace functions are defined as

$$T(v, i, j, \tau) = \text{tr}_{V(g^i)} o(v)\phi_i(g^j)q^{L_0 - c/24}$$

where  $v \in V$  and  $i, j \in \mathbb{Z}_n$ . They transform under  $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$  as

$$T(v, i, j, -1/\tau) = \tau^{\text{wt}[v]} \lambda_{i,j} T(v, j, -i, \tau)$$

where  $\text{wt}[v]$  is the weight of  $v$  with respect to Zhu's second grading and the  $\lambda_{i,j} = \sigma(i, j, S)$  are complex numbers depending only on  $i, j$ .

Combining Theorem 5.1 with the results of [MT] we obtain

### Theorem 5.2

*Up to isomorphism there are exactly  $n^2$  distinct irreducible  $V^G$ -modules, namely  
the eigenspaces  $W^{(i,j)}$ .*

We describe the contragredient module  $W^{(i,j)'} \text{ of } W^{(i,j)}$ .

**Proposition 5.3**

We have

$$W^{(i,j)'} \cong W^{(-i, \alpha(i)-j)}$$

for some function  $\alpha : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$  satisfying  $\alpha(i) = \alpha(-i)$  and  $\alpha(0) = 0$ .

*Proof:* Let  $W$  be an irreducible twisted or untwisted  $V$ -module and let  $\phi_W : G \rightarrow \text{Aut}_{\mathbb{C}}(W)$  denote the action of  $G$  on the vector space  $W$  such that

$$Y_W(gv, z) = \phi_W(g)Y_W(v, z)\phi_W(g)^{-1}$$

for all  $g \in G$ . From the definition of the contragredient module follows

$$Y_{W'}(gv, z) = \phi_{W'}'(g)^{-1}Y_{W'}(v, z)\phi_{W'}'(g)$$

where  $\phi_{W'}' : G \rightarrow \text{Aut}_{\mathbb{C}}(W')$  denotes the dual representation of  $\phi_W$ . By uniqueness this shows that the representation  $\phi_{W'}$  is proportional to the inverse of  $\phi_{W'}'$ . Using  $V(g^i)' \cong V(g^{-i})$  we deduce

$$W^{(i,j)'} \cong W^{(-i, \alpha(i)-j)}.$$

Now  $W'' \cong W$  implies  $j = \alpha(-i) - (\alpha(i) - j)$  so that  $\alpha(i) = \alpha(-i)$ . With  $\phi_0(g) = g$  we have  $W^{(0,0)} = V^G$  so that  $\alpha(0) = 0$ .  $\square$

We determine the fusion algebra  $\mathcal{V}(V^G)$  of  $V^G$ .

**Proposition 5.4**

The  $\mathcal{S}$ -matrix of  $V^G$  is given by

$$\mathcal{S}_{(i,j),(k,l)} = \frac{1}{n}e^{-(il+jk)/n}\lambda_{i,k}.$$

*Proof:* We have

$$\begin{aligned} T(v, i, j, \tau) &= \text{tr}_{V(g^i)} o(v)\phi_i(g^j)q^{L_0-c/24} \\ &= \sum_{k \in \mathbb{Z}_n} \text{tr}_{W^{(i,k)}} o(v)e(jk/n)q^{L_0-c/24} = \sum_{k \in \mathbb{Z}_n} e(jk/n)T_{W^{(i,k)}}(v, \tau) \end{aligned}$$

so that

$$T_{W^{(i,j)}}(v, \tau) = \frac{1}{n} \sum_{l \in \mathbb{Z}_n} e(-lj/n)T(v, i, l, \tau)$$

and

$$\begin{aligned} T_{W^{(i,j)}}(v, S\tau) &= \frac{1}{n} \sum_{k \in \mathbb{Z}_n} e(-jk/n)T(v, i, k, S\tau) \\ &= \frac{1}{n} \sum_{k \in \mathbb{Z}_n} e(-jk/n)\tau^{\text{wt}[v]}\lambda_{i,k}T(v, k, -i, \tau) \\ &= \frac{1}{n} \sum_{k, l \in \mathbb{Z}_n} e(-(jk+il)/n)\tau^{\text{wt}[v]}\lambda_{i,k}T_{W^{(k,l)}}(v, \tau) \end{aligned}$$

by the twisted modular invariance.  $\square$

**Proposition 5.5**

The constants  $\lambda_{i,j}$  satisfy

$$\begin{aligned}\lambda_{i,j} &= \lambda_{j,i} \\ \lambda_{i,j}\lambda_{i,-j} &= e(i\alpha(j)/n) \\ \lambda_{0,i} &= 1\end{aligned}$$

for all  $i, j \in \mathbb{Z}_n$ .

*Proof:* The first equation follows from the symmetry of the  $\mathcal{S}$ -matrix.  $\mathcal{S}^2$  is a permutation matrix sending the index  $(i, j)$  to the index of the contragredient module  $(-i, -j + \alpha(i))$ , i.e.

$$(\mathcal{S}^2)_{(i,j),(l,m)} = \delta_{i,-l}\delta_{j,-m+\alpha(i)}.$$

By the previous proposition we have

$$\begin{aligned}(\mathcal{S}^2)_{(i,j),(l,m)} &= \sum_{a,b \in \mathbb{Z}_n} \mathcal{S}_{(i,j),(a,b)} \mathcal{S}_{(a,b),(l,m)} \\ &= \frac{1}{n^2} \sum_{a,b \in \mathbb{Z}_n} e(-(ib + ja + am + bl)/n) \lambda_{i,a} \lambda_{a,l} \\ &= \frac{1}{n^2} \sum_{a \in \mathbb{Z}_n} e(-a(j+m)/n) \lambda_{i,a} \lambda_{a,l} \sum_{b \in \mathbb{Z}_n} e(-b(i+l)/n) \\ &= \delta_{i,-l} \frac{1}{n} \sum_{a \in \mathbb{Z}_n} e(-a(j+m)/n) \lambda_{i,a} \lambda_{a,l}\end{aligned}$$

so that

$$\frac{1}{n} \sum_{a \in \mathbb{Z}_n} e(-a(j+m)/n) \lambda_{i,a} \lambda_{a,-i} = \delta_{j,\alpha(i)-m}$$

and

$$\frac{1}{n} \sum_{a \in \mathbb{Z}_n} e(-ab/n) \lambda_{i,a} \lambda_{a,-i} = \delta_{b-m,\alpha(i)-m}.$$

Multiplying with  $e(db/n)$  and summing over  $b$  gives the second equation of the proposition. This equation implies  $\lambda_{0,i}^2 = 1$ . In order to prove the last equation we show that  $\lambda_{0,i} \in \mathbb{R}_{\geq 0}$ . We have

$$T(\mathbf{1}, 0, j, -1/\tau) = \lambda_{0,j} T(\mathbf{1}, j, 0, \tau)$$

so that

$$\begin{aligned}\sum_{k=0}^{\infty} e((-1/\tau)(k - c/24)) \operatorname{tr}_{V_k} g^j \\ = \lambda_{0,j} \sum_{k \in \rho_j + (1/n)\mathbb{Z}_{\geq 0}} e((k - c/24)\tau) \dim(V(g^j)_k)\end{aligned}$$

where  $\rho_j \in \mathbb{Q}$  is the conformal weight of  $V(g^j)$  and  $c \in \mathbb{Q}$  the central charge of  $V$ . Specialising to  $\tau = it$  with  $t \in \mathbb{R}_{>0}$  we obtain

$$\begin{aligned} \frac{1}{\lambda_{0,j}} \sum_{k=0}^{\infty} e^{-2\pi k/t} \operatorname{tr}_{V_k} g^j \\ = e^{2\pi c(t-1/t)/24} \sum_{k \in \rho_j + (1/n)\mathbb{Z}_{\geq 0}} e^{-2\pi k t} \dim(V(g^j)_k) \in \mathbb{R}_{\geq 0}. \end{aligned}$$

The limit of the left hand side for  $t \rightarrow 0$  exists and is  $1/\lambda_{0,j}$  because  $V$  is of CFT-type. Hence  $\lambda_{0,j}$  is a non-negative real number.  $\square$

**Proposition 5.6**

*The irreducible  $V^G$ -modules  $W^{(i,j)}$  are simple currents.*

*Proof:* As before let  $(i,j)'$  denote the index of the contragredient module of  $W^{(i,j)}$ . Then

$$\begin{aligned} \mathcal{S}_{(i,j),(k,l)} \mathcal{S}_{(i,j)',(k,l)} \\ = \mathcal{S}_{(i,j),(k,l)} \mathcal{S}_{(-i,\alpha(i)-j),(k,l)} \\ = \frac{1}{n} e(-(kj+il)/n) \lambda_{i,k} \frac{1}{n} e(-(k(\alpha(i)-j) + (-i)l)/n) \lambda_{-i,k} \\ = \frac{1}{n^2} e(-k\alpha(i)/n) \lambda_{i,k} \lambda_{-i,k} \\ = \frac{1}{n^2} \end{aligned}$$

by Proposition 5.5. We compute the fusion coefficients with the Verlinde formula

$$\begin{aligned} N_{(i,j),(i,j)}^{(l,k)} &= \sum_{a,b \in \mathbb{Z}_n} \frac{\mathcal{S}_{(i,j),(a,b)} \mathcal{S}_{(i,j)',(a,b)} \mathcal{S}_{(a,b),(l,k)'}}{\mathcal{S}_{(0,0),(a,b)}} \\ &= \frac{1}{n^2} \sum_{a,b \in \mathbb{Z}_n} \frac{\mathcal{S}_{(a,b),(l,k)'}}{\mathcal{S}_{(0,0),(a,b)}} \\ &= \frac{1}{n^2} \sum_{a,b \in \mathbb{Z}_n} \frac{\mathcal{S}_{(a,b),(-l,\alpha(l)-k)}}{\mathcal{S}_{(0,0),(a,b)}} \\ &= \frac{1}{n^2} \sum_{a,b \in \mathbb{Z}_n} \frac{e(-((-l)b + a(\alpha(l)-k))/n) \lambda_{a,-l}}{\lambda_{0,a}} \\ &= \frac{1}{n^2} \sum_{a \in \mathbb{Z}_n} e(a(k - \alpha(l))/n) \lambda_{a,-l} \sum_{b \in \mathbb{Z}_n} e(lb/n) \\ &= \delta_{l,0} \frac{1}{n} \sum_{a \in \mathbb{Z}_n} e(a(k - \alpha(0))/n) \lambda_{a,0} \\ &= \delta_{l,0} \frac{1}{n} \sum_{a \in \mathbb{Z}_n} e(ak/n) \\ &= \delta_{l,0} \delta_{k,0}. \end{aligned}$$

This means

$$W^{(i,j)} \boxtimes W^{(i,j)'} \cong W^{(0,0)}$$

for all  $i, j \in \mathbb{Z}_n$ . From this we derive that all  $W^{(i,j)}$  are simple currents.  $\square$

The proposition implies that the fusion algebra of  $V^G$  is the group algebra  $\mathbb{C}[D]$  of a finite abelian group  $D$  of order  $n^2$ . Propositions 5.4 and 5.5 show

$$\mathcal{S}_{(0,0),(0,0)} = \mathcal{S}_{(0,0),(i,j)} = \frac{1}{\sqrt{|D|}}.$$

Since the positivity assumption in the results of Section 3 only enters through Proposition 3.2 this implies that all results of Section 3 hold for  $V^G$  without the assumption on the conformal weights. In particular the reduction of the conformal weights modulo 1 defines a quadratic form  $q_\Delta$  on  $D$  whose associated bilinear form  $b_\Delta$  is non-degenerate.

**Proposition 5.7**

*The fusion product takes the form*

$$W^{(i,j)} \boxtimes W^{(k,l)} \cong W^{(i+k,j+l+c(i,k))}$$

for some symmetric, normalised 2-cocycle  $c : \mathbb{Z}_n \times \mathbb{Z}_n \rightarrow \mathbb{Z}_n$  satisfying

$$e(-ac(i,k)/n) = \frac{\lambda_{i,a}\lambda_{k,a}}{\lambda_{i+k,a}}$$

for all  $i, k, a \in \mathbb{Z}_n$ .

*Proof:* Let  $W^{(i,j)} \boxtimes W^{(k,l)} \cong W^{(s,t)}$ . Then

$$\mathcal{S}_{(i,j),(a,b)} \mathcal{S}_{(k,l),(a,b)} = \frac{1}{n} \mathcal{S}_{(s,t),(a,b)}$$

for all  $a, b \in \mathbb{Z}_n$ . This implies

$$\lambda_{i,a}\lambda_{k,a}/\lambda_{s,a} = e(-(sb+ta)/n)e((ib+ja)/n)e((kb+la)/n).$$

Taking  $a = 0$  we obtain  $s = i + k \pmod n$  because  $\lambda_{i,0} = 1$ . For  $b = 0$  we get

$$\lambda_{i,a}\lambda_{k,a}/\lambda_{i+k,a} = e((j+l-t)a/n).$$

This shows that  $t-j-l$  depends only on  $i$  and  $k$  and we define  $c(i,k) = t-j-l$ . The associativity of the fusion algebra  $\mathcal{V}(V^G)$  implies that  $e(-c(i,k)/n)$  is a 2-cocycle  $\mathbb{Z}_n \times \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ . This cocycle is symmetric since  $\mathcal{V}(V^G)$  is commutative and normalised since  $\lambda_{0,1} = 1$ .  $\square$

The 2-cocycle  $c$  is related to the map  $\alpha$  by

$$\alpha(i) + c(i, -i) = 0 \pmod n.$$



The maps  $\mathbb{Z}_n \rightarrow D$ ,  $j \mapsto (0, j)$  and  $D \rightarrow \mathbb{Z}_n$ ,  $(i, j) \mapsto i$  give an exact sequence

$$0 \rightarrow \mathbb{Z}_n \rightarrow D \rightarrow \mathbb{Z}_n \rightarrow 0$$

i.e.  $D$  is a central extension of  $\mathbb{Z}_n$  by  $\mathbb{Z}_n$ . This extension is determined up to isomorphism by the cohomology class of the 2-cocycle  $c$  in  $H^2(\mathbb{Z}_n, \mathbb{Z}_n)$ . The group  $H^2(\mathbb{Z}_n, \mathbb{Z}_n)$  is isomorphic to  $\mathbb{Z}_n$  and the 2-cocycle  $c_d$  corresponding to  $d$  in  $\mathbb{Z}_n$  is represented by

$$c_d(i, j) = \begin{cases} 0 & \text{if } i_n + j_n < n \\ d & \text{if } i_n + j_n \geq n \end{cases}$$

where  $i_n$  denotes a representative of  $i$  in  $\{0, \dots, n-1\}$ . The cohomology class  $d$  of  $c$  can be determined by

$$d = c(1, 1) + c(1, 2) + \dots + c(1, n-1) \pmod{n}.$$

We write  $\rho_i$  for the conformal weight  $\rho(V(g^i))$  of the irreducible  $g^i$ -twisted  $V$ -module  $V(g^i)$ .

**Proposition 5.8**

We can choose the representation  $\phi_i$  of  $G$  on  $V(g^i)$  such that

$$\phi_i(g)^{(i, n)} = e((i/(i, n))^{-1}(L_0 - \rho_i))$$

where  $(i/(i, n))^{-1}$  denotes the inverse of  $i/(i, n)$  modulo  $n/(i, n)$ . Then the irreducible  $V^G$ -modules  $W^{(i, j)}$  have conformal weights

$$\rho(W^{(i, j)}) = \rho_i + ij/n \pmod{1}$$

and the function  $\alpha$  satisfies

$$i\alpha(i) = 0 \pmod{n}.$$

*Proof:* We prove the first statement in the case  $(i, n) = 1$ . The general case can be derived from this. The map

$$\phi_i(g^j) = e(i^{-1}j(L_0 - \rho_i))$$

satisfies

$$\phi_i(g^j)Y_{V(g^i)}(v, z)\phi_i(g^j)^{-1}w = Y_{V(g^i)}(g^jv, z)w$$

for all  $v \in V$  and  $w \in V(g^i)$  which by uniqueness means that it is a possible choice for the representation  $\phi_i$ . This implies the first statement. The second statement follows easily from the first. For the third statement observe that  $\rho_i = \rho_{-i}$ .  $\square$

If we chose the maps  $\phi_i$  as in the proposition then the quadratic form  $q_\Delta$  on  $D$  is given by

$$q_\Delta((i, j)) = \rho_i + ij/n \pmod{1}$$

and

$$b_{\Delta}((i, j), (k, l)) = \rho_{i+k} - \rho_i - \rho_k + \frac{il + jk}{n} + \frac{(i+k)c(i, k)}{n} \pmod{1}.$$

We will use this choice in the proofs of the next 3 results.

**Proposition 5.9**

*The cohomology class of the central extension defined by the 2-cocycle  $c$  is given by*

$$d = 2n^2 \rho_1 \pmod{n}.$$

*Proof:* We have  $q_{\Delta}((i, j)) = \rho_i + ij/n \pmod{1}$  so that

$$\lambda_{i,i} = ne(2ij/n) \mathcal{S}_{(i,j),(i,j)} = e(2ij/n) e(-2q_{\Delta}((i, j))) = e(-2\rho_i).$$

Then

$$\begin{aligned} e(d/n) &= e((c(1, 1) + c(1, 2) + \dots + c(1, n-1))/n) \\ &= \frac{\lambda_{2,1}}{\lambda_{1,1}\lambda_{1,1}} \frac{\lambda_{3,1}}{\lambda_{1,1}\lambda_{2,1}} \cdot \dots \cdot \frac{\lambda_{n,1}}{\lambda_{1,1}\lambda_{n-1,1}} \\ &= \frac{\lambda_{n,1}}{\lambda_{1,1}^n} = \lambda_{1,1}^{-n} = e(2n\rho_1) \end{aligned}$$

because  $\lambda_{n,1} = \lambda_{0,1} = 1$ . □

Since  $D$  has order  $n^2$  the bilinear form  $b_{\Delta}$  takes its values in  $(1/n^2)\mathbb{Z}/\mathbb{Z}$  and the associated quadratic form  $q_{\Delta}$  in  $(1/2n^2)\mathbb{Z}/\mathbb{Z}$ . We show that the values of  $q_{\Delta}$  actually lie in  $(1/n^2)\mathbb{Z}/\mathbb{Z}$ .

**Theorem 5.10**

*The unique irreducible  $g$ -twisted  $V$ -module  $V(g)$  has conformal weight*

$$\rho_1 \in (1/n^2)\mathbb{Z}$$

*and more generally  $V(g^i)$  has conformal weight  $\rho_i \in ((n, i)^2/n^2)\mathbb{Z}$ .*

*Proof:* It is sufficient to prove the statement for  $i = 1$ . We have

$$n^2 \rho_1 = n^2 q_{\Delta}((1, j)) = q_{\Delta}(n(1, j)) = q_{\Delta}((0, k)) \pmod{1}$$

for some  $k \in \mathbb{Z}_n$ . But this last value is 0  $\pmod{1}$ . This proves the theorem. □

This result generalises Theorem 1.6 (i) in [DLM]. The value of  $\rho_1$  determines the group structure of  $D$ . We will see that it also determines the quadratic form  $q_{\Delta}$  up to isomorphism.

We define the type  $t \in \mathbb{Z}_n$  of  $g$  by

$$t = n^2 \rho_1 \pmod{n},$$

i.e.  $d = 2t \pmod{n}$ . Let  $N$  be the smallest positive multiple of  $n$  such that  $N\rho_1 = 0 \pmod{1}$ , i.e.  $N = n^2/(t, n)$ . Then  $N$  is the level of  $q_{\Delta}$ .

**Proposition 5.11**

The conformal weights  $\rho_i$  satisfy

$$\rho_i = \frac{i^2 t}{n^2} \mod \frac{(i, n)}{n}.$$

*Proof:* We have

$$\begin{aligned} i^2 \rho_1 &= i^2 q_\Delta((1, 0)) = q_\Delta(i(1, 0)) \\ &= q_\Delta((i, c(1, 1) + \dots + c(1, i-1))) \\ &= \rho_i + \frac{i(c(1, 1) + \dots + c(1, i-1))}{n} \end{aligned}$$

so that

$$i^2 \rho_1 = \rho_i \mod \frac{(i, n)}{n}.$$

On the other hand  $i^2 \rho_1 = i^2 t/n^2 \mod (i, n)/n$ . This proves the proposition.  $\square$

The 2-cocycles  $c$  and  $c_d$  with  $d = 2t \mod n$  differ by a coboundary. Shifting  $c$  by a coboundary corresponds to redefining the maps  $\phi_i$ . Standard arguments show

**Theorem 5.12**

Let  $V$  be a simple, rational,  $C_2$ -cofinite, holomorphic vertex operator algebra of CFT-type and  $G = \langle g \rangle$  a finite, cyclic group of automorphisms of  $V$  of order  $n$ . Suppose  $g$  has type  $t \mod n$  and let  $d = 2t \mod n$ . Then we can define the irreducible  $V^G$ -modules  $W^{(i, j)}$  or equivalently the  $\phi_i$  such that

- i)  $W^{(i, j)} \boxtimes W^{(k, l)} \cong W^{(i+k, j+l+c_d(i, k))}$
- ii)  $W^{(i, j)}$  has conformal weight  $q_\Delta((i, j)) = \frac{ij}{n} + \frac{i^2 t_n}{n^2} \mod 1$
- iii)  $W^{(i, j)'} \cong W^{(-i, -j-c_d(i, -i))}$ ,
- iv)  $\mathcal{S}_{(i, j), (k, l)} = \frac{1}{n} e(-(il + jk)/n) \lambda_{i, k} = \frac{1}{n} e(-(il + jk)/n) e(-2t_n i_n k_n/n^2)$   
i.e.  $\lambda_{i, k} = e(-2t_n i_n k_n/n^2)$

for  $i, j, k, l \in \mathbb{Z}_n$ .

We remark that the maps  $\phi_i$  chosen in the theorem are not necessarily the same as in Proposition 5.8.

The fusion group  $D$  is given as a set by

$$D = \mathbb{Z}_n \times \mathbb{Z}_n$$

with multiplication

$$(i, j) + (k, l) = (i + k, j + l + c_d(i, k)).$$

This group is isomorphic to the group

$$\mathbb{Z}_{n^2/(n, d)} \times \mathbb{Z}_{(n, d)}.$$

**Proposition 5.13**

The discriminant form  $D$  is isomorphic to the discriminant form of the even lattice with Gram matrix  $\begin{pmatrix} -2t_n & n \\ n & 0 \end{pmatrix}$ .

For homogeneous  $v \in V^G$  the functions  $T_{W^{(i,j)}}(v, \tau)$ ,  $i, j \in \mathbb{Z}_n$  transform under Zhu's representation  $\rho_{V^G}$ . Up to a character this representation is the Weil representation  $\rho_D$ . Since  $V$  is holomorphic the central charge  $c$  of  $V$  is a positive integer satisfying  $c \equiv 0 \pmod{8}$ . In particular  $c$  is even so that the Weil representation  $\rho_D$  is a representation of  $\mathrm{SL}_2(\mathbb{Z})$  and

$$\rho_{V^G}(M) = \chi_c(M) \rho_D(M)$$

for all  $M \in \mathrm{SL}_2(\mathbb{Z})$ . The character  $\chi_c$  is defined by

$$\chi_c(S) = e(c/8) = 1 \quad \text{and} \quad \chi_c(T) = e(-c/24).$$

The Weil representation is trivial on  $\Gamma(N)$  where  $N$  is the level of  $D$  so that we obtain

**Theorem 5.14**

Let  $V$  be a simple, rational,  $C_2$ -cofinite, holomorphic vertex operator algebra of CFT-type and central charge  $c$ . Let  $G = \langle g \rangle$  be a finite, cyclic group of automorphisms of  $V$  of order  $n$ . Suppose the fusion group  $D$  of  $V^G$  has level  $N$  under  $q_\Delta$ . Then the trace functions  $T_{W^{(i,j)}}(v, \tau)$  and the twisted traces  $T(v, i, j, \tau)$  are modular forms of weight  $\mathrm{wt}[v]$  for a congruence subgroup of  $\mathrm{SL}_2(\mathbb{Z})$  of level  $N$  if  $24|c$  and level  $\mathrm{lcm}(3, N)$  otherwise.

This result generalises Theorem 1.6 (ii) in [DLM].

Combining our knowledge of the fusion algebra of  $V^G$  with the results about abelian intertwining algebras we can construct new holomorphic vertex operator algebras.

**Theorem 5.15**

Let  $V$  be a simple, rational,  $C_2$ -cofinite, holomorphic vertex operator algebra of CFT-type and central charge  $c$ . Let  $G = \langle g \rangle$  be a finite, cyclic group of automorphisms of  $V$  of order  $n$ . Suppose the modules  $W^{(i,j)}$ ,  $(i, j) \neq (0, 0)$  of  $V^G$  have positive conformal weights. Then the direct sum

$$W = \bigoplus_{i,j \in \mathbb{Z}_n} W^{(i,j)} = \bigoplus_{\gamma \in D} W^\gamma$$

has the structure of an abelian intertwining algebra extending the vertex operator algebra structure of  $V^G$  with associated finite quadratic space  $(D, -q_\Delta)$ . Let  $H$  be an isotropic subgroup of  $D$ . Then

$$W^H = \bigoplus_{\gamma \in H} W^\gamma$$

admits the structure of a simple, rational,  $C_2$ -cofinite, self-contragredient vertex operator algebra of CFT-type extending the vertex operator algebra structure of  $V^G$ . If  $H = H^\perp$  then  $W^H$  is holomorphic.

We make some comments on the theorem.

We call  $W^H$  the orbifold of  $V$  with respect to  $G$  and  $H$  and denote it by  $V^{\text{orb}(G,H)}$ . Any isotropic subgroup  $H$  of  $D$  of order  $n$  satisfies  $H = H^\perp$ .

If  $g$  is of type  $t \neq 0 \pmod n$  then  $W^H$  can already be obtained from some lower order automorphism with  $t = 0 \pmod n$ .

The case when  $g$  is of type  $t = 0 \pmod n$  is particularly nice. Then it is possible to choose the representations  $\phi_i$  such that

- i)  $W^{(i,j)} \boxtimes W^{(k,l)} \cong W^{(i+k,j+l)}$
- ii)  $W^{(i,j)}$  has conformal weight  $q_\Delta((i,j)) = ij/n \pmod 1$
- iii)  $W^{(i,j)'} \cong W^{(-i,-j)}$
- iv)  $\mathcal{S}_{(i,j),(k,l)} = \frac{1}{n}e(-(jk+il)/n)$ , i.e.  $\sigma(i,k,S) = \lambda_{i,k} = 1$
- v)  $\mathcal{T}_{(i,j),(k,l)} = e(ij/n - c/24)\delta_{(i,j),(k,l)}$ , i.e.  $\sigma(i,k,T) = e(-c/24)$

for  $i, j, k, l \in \mathbb{Z}_n$ . This means that the fusion algebra of  $V^G$  is the group algebra of the abelian group  $\mathbb{Z}_n \times \mathbb{Z}_n$  with quadratic form

$$q_\Delta((i,j)) = \frac{ij}{n} \pmod 1.$$

The direct sum of irreducible  $V^G$ -modules

$$W = \bigoplus_{i,j \in \mathbb{Z}_n} W^{(i,j)}$$

has the structure of an abelian intertwining algebra and the sum

$$W^H = \bigoplus_{i \in \mathbb{Z}_n} W^{(i,0)}$$

over the maximal isotropic subgroup  $H = \{(i,0) \mid i \in \mathbb{Z}_n\}$  is a simple, rational,  $C_2$ -cofinite, holomorphic vertex operator algebra of CFT-type which extends the vertex operator algebra structure on  $V^G$ . In this case we simply write  $V^{\text{orb}(G)}$  for the orbifold  $W^H$ . We can define an automorphism  $k$  of  $V^{\text{orb}(G)}$  of order  $n$  by setting  $kv = e(i/n)v$  for  $v \in W^{(i,0)}$ . Let  $K$  be the cyclic group generated by  $k$ . Then  $(V^{\text{orb}(G)})^K = W^{(0,0)} = V^G$  and the twisted modules of  $V^{\text{orb}(G)}$  corresponding to  $k^j$  are given by  $V^{\text{orb}(G)}(k^j) = \bigoplus_{i \in \mathbb{Z}_n} W^{(j,i)}$ . Hence

$$(V^{\text{orb}(G)})^{\text{orb}(K)} = \bigoplus_{i \in \mathbb{Z}_n} W^{(0,i)} = V,$$

i.e. there is an inverse orbifold which gives back  $V$ .

## 6 Schellekens' list

In this section we show that Schellekens' classification of  $V_1$ -structures of meromorphic conformal field theories of central charge 24 [S] is a rigorous theorem on vertex operator algebras.

Let  $g$  be a finite-dimensional simple Lie algebra over  $\mathbb{C}$  and  $h$  a Cartan subalgebra of  $g$ . We fix a system  $\Phi^+$  of positive roots and denote by  $\theta$  the corresponding highest root. We normalise the non-degenerate invariant symmetric bilinear form on  $g$  such that  $(\theta, \theta) = 2$ . This form is related to the Killing form  $k(\cdot, \cdot)$  on  $g$  by  $(\cdot, \cdot) = 2h^\vee k(\cdot, \cdot)$  where  $h^\vee$  is the dual Coxeter number of  $g$ . Note that  $\dim(g) > (h^\vee/2)^2$ . The affine Lie algebra  $\hat{g}$  associated to  $g$  is defined as

$$\hat{g} = g \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$$

where  $K$  is central and

$$[a(m), b(n)] = [a, b](m+n) = m\delta_{m+n}(a, b)K$$

with  $a(m) = a \otimes t^m$ . Define  $\hat{g}_+ = g \otimes t\mathbb{C}[t]$ ,  $\hat{g}_- = g \otimes t^{-1}\mathbb{C}[t^{-1}]$  and identify  $g$  with  $g \otimes 1$ . Then

$$\hat{g} = \hat{g}_+ \oplus \hat{g}_- \oplus g \oplus \mathbb{C}K.$$

Let  $V$  be a  $g$ -module and  $k \in \mathbb{C}$ . We can extend the action of  $g$  to an action of  $\hat{g}_+ \oplus g \oplus \mathbb{C}K$  by letting  $\hat{g}_+$  act trivially and  $K$  as  $k \text{Id}$ . Then we form the induced  $\hat{g}$ -module

$$\hat{V}_k = U(\hat{g}) \otimes_{U(\hat{g}_+ \oplus g \oplus \mathbb{C}K)} V.$$

Applied to an irreducible highest-weight module  $L(\lambda)$  of  $g$  we obtain the  $\hat{g}$ -module  $\hat{L}(\lambda)_k$  which we denote by  $M_{k,\lambda}$ . If  $k \neq -h^\vee$  then  $M_{k,0}$  is a vertex operator algebra of central charge  $k \dim(g)/(k + h^\vee)$ . The first homogeneous pieces of  $M_{k,0}$  are given by  $\mathbb{C}$ ,  $g$ ,  $g \oplus \wedge^2(g)$ , etc. Let  $J_{k,\lambda}$  be the maximal proper  $\hat{g}$ -submodule of  $M_{k,\lambda}$ . Then the quotient  $L_{k,\lambda} = M_{k,\lambda}/J_{k,\lambda}$  is an irreducible  $\hat{g}$ -module. If  $k$  is a positive integer  $J_{k,0}$  is generated by  $(e_\theta)(-1)^{k+1}\mathbf{1}$  where  $e_\theta$  is any non-zero element in the root space  $g_\theta$  of  $g$  and  $L_{k,0}$  is a rational vertex operator algebra whose irreducible modules are the spaces  $L_{k,\lambda}$  where  $\lambda \in h^*$  ranges over the integrable weights satisfying  $(\lambda, \theta) \leq k$  (cf. [FZ], Theorem 3.1.3). The character  $\chi_{L_{k,\lambda}} : H \times h \rightarrow \mathbb{C}$  of  $L_{k,\lambda}$  is defined by

$$\chi_{L_{k,\lambda}}(\tau, z) = \text{tr}_{L_{k,\lambda}} e^{2\pi i z} q^{L_0 - c/24}.$$

Let  $V$  be a simple, rational,  $C_2$ -cofinite, self-contragredient vertex operator algebra of CFT-type. Then  $V$  carries a unique symmetric, invariant, bilinear form  $\langle \cdot, \cdot \rangle$  satisfying  $\langle \mathbf{1}, \mathbf{1} \rangle = -1$  where  $\mathbf{1}$  denotes the vacuum of  $V$  and the subspace  $V_1$  of  $L_0$ -degree 1 is a Lie algebra under  $[a, b] = a_0 b$ . The bilinear form  $\langle \cdot, \cdot \rangle$  is non-degenerate on  $V$  and on  $V_1$ .

Now we assume in addition that  $V$  is holomorphic and of central charge 24. Then  $V_1$  is either trivial, abelian of dimension 24 or semisimple (cf. [S] and

[DM1]). In the second case  $V$  is isomorphic to the vertex algebra of the Leech lattice. We consider the third case that  $V_1$  is a semisimple Lie algebra

$$g = g_{1,k_1} \oplus \dots \oplus g_{n,k_n}.$$

Then the restriction of  $\langle, \rangle$  to  $g_i$  satisfies

$$\langle, \rangle = k_i(\cdot, \cdot)$$

where  $(\cdot, \cdot)$  is the normalised bilinear form on  $g_i$  and  $k_i$  a positive integer [DM2]. The map  $a_i(m) \mapsto (a_i)_m$  sending  $a_i \in g_i$  to its  $m$ -th mode defines a representation of  $\hat{g}_i$  on  $V$  of level  $k_i$ . The vertex operator subalgebra of  $V$  generated by  $V_1$  is isomorphic to

$$L_{k_1,0} \otimes \dots \otimes L_{k_n,0}$$

and the Virasoro elements of both vertex operator algebras coincide. Since  $L_{k_1,0} \otimes \dots \otimes L_{k_n,0}$  is rational and the  $L_0$ -eigenspaces of  $V$  are finite-dimensional,  $V$  decomposes into finitely many irreducible  $L_{k_1,0} \otimes \dots \otimes L_{k_n,0}$ -modules

$$V = \bigoplus_{(\lambda_1, \dots, \lambda_n)} m_{(\lambda_1, \dots, \lambda_n)} L_{k_1, \lambda_1} \otimes \dots \otimes L_{k_n, \lambda_n}$$

where each  $\lambda_i$  is an integrable weight satisfying  $(\lambda_i, \theta_i) \leq k_i$ . The Lie algebra  $g$  acts on  $V_n$  by  $(a, v) \mapsto a_0 v$ . Let

$$h = h_{1,k_1} \oplus \dots \oplus h_{n,k_n}$$

be a Cartan subalgebra of  $g$ . Then the character

$$\chi_V(\tau, z) = \text{tr}_V e^{2\pi i z_0} q^{L_0 - 1}$$

is holomorphic on  $H \times h$  and satisfies

$$\chi_V \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = \exp \left( 2\pi i \frac{\langle z, z \rangle}{2} \frac{c}{c\tau + d} \right) \chi_V(\tau, z)$$

(cf. [KM]). A finite-dimensional  $g$ -module  $M$  decomposes into weight spaces and we define for  $z \in h$  the function

$$S_M^j(z) = \sum_{\mu \in \Pi(M)} m_\mu \mu(z)^j$$

where  $\Pi(M)$  denotes the set of weights of  $M$  and  $m_\mu$  the multiplicity of  $\mu$ . For example  $S_M^0(z) = \dim(M)$ .

### Theorem 6.1

We have

$$S_{V_1}^2(z) = \frac{1}{12}(\dim(V_1) - 24)\langle z, z \rangle$$

and for  $V_2$

$$\begin{aligned}
S_{V_2}^2(z) &= 32808 \langle z, z \rangle - 2 \dim(V_1) \langle z, z \rangle \\
S_{V_2}^4(z) &= 240 S_{V_1}^4(z) + 15264 \langle z, z \rangle^2 - \dim(V_1) \langle z, z \rangle^2 \\
S_{V_2}^6(z) &= -504 S_{V_1}^6(z) + 900 S_{V_1}^4(z) \langle z, z \rangle + 11160 \langle z, z \rangle^3 - 15 \dim(V_1) \langle z, z \rangle^3 \\
S_{V_2}^8(z) &= 480 S_{V_1}^8(z) - 2352 S_{V_1}^6(z) \langle z, z \rangle + 2520 S_{V_1}^4(z) \langle z, z \rangle^2 + 10920 \langle z, z \rangle^4 \\
&\quad - 35 \dim(V_1) \langle z, z \rangle^4 \\
S_{V_2}^{10}(z) &= -264 S_{V_1}^{10}(z) + 2700 S_{V_1}^8(z) \langle z, z \rangle - 7560 S_{V_1}^6(z) \langle z, z \rangle^2 \\
&\quad + 6300 S_{V_1}^4(z) \langle z, z \rangle^3 + 13230 \langle z, z \rangle^5 - \frac{315}{4} \dim(V_1) \langle z, z \rangle^5
\end{aligned}$$

and

$$\begin{aligned}
48 S_{V_2}^{14}(z) - 364 S_{V_2}^{12}(z) \langle z, z \rangle &= -1152 S_{V_1}^{14}(z) \\
&\quad + 288288 S_{V_1}^{10}(z) \langle z, z \rangle^2 - 2162160 S_{V_1}^8(z) \langle z, z \rangle^3 + 5045040 S_{V_1}^6(z) \langle z, z \rangle^4 \\
&\quad - 3783780 S_{V_1}^4(z) \langle z, z \rangle^5 - 5405400 \langle z, z \rangle^7 + 45045 \dim(V_1) \langle z, z \rangle^7.
\end{aligned}$$

*Proof:* Let

$$P(\tau, z) = \exp \left( -(2\pi i)^2 \frac{\langle z, z \rangle}{24} E_2(\tau) \right) \Delta(\tau) \chi_V(\tau, z)$$

where

$$\Delta(\tau) = q \prod_{m=1}^{\infty} (1 - q^m)^{24}$$

is Dedekind's  $\Delta$ -function and

$$E_2(\tau) = 1 - 24 \sum_{m=1}^{\infty} \sigma(m) q^m$$

the Eisenstein series of weight 2. Since  $\Delta$  is a modular form of weight 12 and  $E_2$  transforms as

$$E_2 \left( \frac{a\tau + b}{c\tau + d} \right) = E_2(\tau) (c\tau + d)^2 + \frac{12}{2\pi i} c(c\tau + d)$$

for  $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$  we have

$$P \left( \frac{a\tau + b}{c\tau + d}, \frac{z}{c\tau + d} \right) = (c\tau + d)^{12} P(\tau, z).$$

This implies that the  $m$ -th coefficient in the Taylor expansion of  $P$  in  $z$  is a modular form for  $\text{SL}_2(\mathbb{Z})$  of weight  $12 + m$ . Using

$$\begin{aligned}
\chi_V(\tau, z) &= q^{-1} + \sum_{\lambda \in \Pi(V_1)} m_{\lambda} e^{2\pi i \lambda(z)} + q \sum_{\lambda \in \Pi(V_2)} m_{\lambda} e^{2\pi i \lambda(z)} + \dots \\
&= q^{-1} + \sum_{m=0}^{\infty} \frac{(2\pi i)^m}{m!} S_{V_1}^m(z) + q \sum_{m=0}^{\infty} \frac{(2\pi i)^m}{m!} S_{V_2}^m(z) + \dots
\end{aligned}$$



we find that the coefficient of degree 2 in the Taylor expansion of  $P(\tau, z)$  is  $(2\pi i)^2$  times

$$\begin{aligned} & -\frac{\langle z, z \rangle}{24} + q \left( \frac{1}{2} S_{V_1}^2(z) - \frac{\langle z, z \rangle}{24} (\dim(V_1) - 48) \right) \\ & + q^2 \left( \frac{1}{2} S_{V_2}^2(z) - 12 S_{V_1}^2(z) - \dim(V_2) \frac{\langle z, z \rangle}{24} + 2 \dim(V_1) \langle z, z \rangle - \frac{63}{2} \langle z, z \rangle \right) \\ & + \dots \end{aligned}$$

Since the space of modular forms for  $\mathrm{SL}_2(\mathbb{Z})$  of weight 14 is spanned by

$$E_{14}(\tau) = 1 - 24q - 196632q^2 - 38263776q^3 + \dots$$

this implies

$$\langle z, z \rangle = \frac{1}{2} S_{V_1}^2(z) - \frac{\langle z, z \rangle}{24} (\dim(V_1) - 48)$$

which gives the first equation in the theorem. Comparing the coefficients at  $q^0$  and  $q^2$  and using  $\dim(V_2) = 196884$  we obtain the second equation of the theorem. Looking at higher orders in  $z$  and using the relations between the coefficients of modular forms we can derive the other equations.  $\square$

The following result is well-known (cf. [S], [DM1]).

### Corollary 6.2

For each simple component  $g_i$  of  $g$  we have

$$\frac{h_i^\vee}{k_i} = \frac{\dim(g) - 24}{24}.$$

*Proof:* The first equation of Theorem 6.1 gives

$$\frac{1}{12} (\dim(g) - 24) \langle z, z \rangle = \sum_{\mu \in \Phi} \mu(z)^2$$

where  $\Phi$  is the root system of  $g$ . Restricting  $z$  to  $z_i \in h_i$  we obtain

$$\frac{1}{12} (\dim(g) - 24) k_i(z_i, z_i) = \sum_{\mu \in \Phi_i} \mu_i(z_i)^2.$$

An elementary calculation shows

$$\sum_{\mu \in \Phi_i} \mu_i(z_i)^2 = 2h_i^\vee(z_i, z_i).$$

This proves the statement.  $\square$

Now we show how Theorem 6.1 and Corollary 6.2 can be used to classify the possible affine structures of  $V$ .

**Proposition 6.3**

The equation  $h_i^\vee/k_i = (\dim(g) - 24)/24$  has 221 solutions.

*Proof:* The equation gives the following inequality

$$\dim(g) = \sum_{i=1}^n \dim(g_i) > \frac{1}{4} \sum_{i=1}^n (h_i^\vee)^2 = \frac{1}{4} \left( \frac{\dim(g) - 24}{24} \right)^2 \sum_{i=1}^n k_i^2.$$

Since

$$\sum_{i=1}^n k_i^2 \geq 1$$

this implies  $\dim(g) \leq 2352$ . There is a finite set of semisimple Lie algebras satisfying this condition and a computer search yields from this set a list of 221 solutions.  $\square$

Let  $V$  be as above with  $V_1$  given by one of the solutions

$$g = g_{1,k_1} \oplus \dots \oplus g_{n,k_n}$$

of  $h_i^\vee/k_i = (\dim(g) - 24)/24$ . Write  $L_{k,\lambda}$  for  $L_{k_1,\lambda_1} \otimes \dots \otimes L_{k_n,\lambda_n}$ . Then  $V$  decomposes as  $L_{k,\lambda}$ -module as

$$V = \bigoplus_{\lambda} m_{\lambda} L_{k,\lambda}$$

where  $\lambda_i$  ranges over the integrable weights of  $g_i$  satisfying  $(\lambda_i, \theta_i) \leq k_i$ . Since  $V_0 = \mathbb{C}\mathbf{1}$  and  $V_1 = g$ , the modules  $L_{k,\lambda}$  with  $\lambda \neq 0$  appearing in the decomposition have conformal weight at least 2. In particular

$$V_2 = (L_{k,0})_2 \oplus \bigoplus_{\lambda} m_{\lambda} (L_{k,\lambda})_2$$

where the sum extends over the  $\lambda \neq 0$  such that  $L_{k,\lambda}$  has conformal weight 2. Recall that the structure of  $(L_{k,0})_2$  is well-known. We write

$$S_{V_2}^j = S_{(L_{k,0})_2}^j + \sum_{\lambda} m_{\lambda} S_{(L_{k,\lambda})_2}^j$$

and plug this decomposition into the second set of equations in Theorem 6.1. Introducing coordinates on  $h$  we obtain a large system of linear equations for the  $m_{\lambda}$  which by the existence of  $V$  has a solution in the non-negative integers. We can reduce the number of variables  $m_{\lambda}$  by identifying the modules  $L_{k,\lambda}$  which become isomorphic under exchange of isomorphic simple components of  $g$ . The resulting reduced system still has a solution in  $\mathbb{Z}_{\geq 0}$ .

**Theorem 6.4**

Let  $V$  be a simple, rational,  $C_2$ -cofinite, holomorphic vertex operator algebra of CFT-type and central charge 24. Then either  $V_1 = 0$ ,  $\dim(V_1) = 24$  and  $V$  is isomorphic to the lattice vertex operator algebra of the Leech lattice or  $V_1$  is one of 69 semisimple Lie algebras described in Table 1 of [S].

*Proof:* If  $V_1$  is non-abelian then it is one of the 221 solutions of the condition  $h_i^\vee/k_i = (\dim(g) - 24)/24$ . For 69 of these Schellekens [S] gives an explicit candidate decomposition  $V = \bigoplus_\lambda m_\lambda L_{k,\lambda}$ . We verified in these cases that the multiplicities satisfy the reduced system described above. It remains to eliminate the other solutions. The case  $A_{3,4}^2 A_{1,2}^6$  will be treated separately below. Of the 151 remaining systems 140 possess no solution in  $\mathbb{Q}_{\geq 0}$ . A further 4 of the systems have no solution in  $\mathbb{Z}$ . This leaves 7 systems for which we must rule out the existence of a solution in  $\mathbb{Z}_{\geq 0}$ . We compute a rational upper bound on each variable  $m_\lambda$ . If this upper bound is less than 1 then  $m_\lambda$  should in fact vanish. We add this equation to the reduced system. This excludes the systems  $A_{3,16} A_{1,8}^5$ ,  $A_{1,8}^2 A_{2,12}^3$ ,  $A_{1,6}^6 G_{2,12}$ ,  $A_{2,9}^4$  and  $A_{1,4}^2 C_{2,6}^3$ . The two remaining Lie algebras are  $A_{1,16}^9$  and  $A_{1,8}^{10}$ . In the associated reduced systems we isolate a subset of variables possessing small upper bounds and assign integer values to these variables within their ranges. All possibilities lead to systems that have no solution in  $\mathbb{Q}$ .

In the case  $A_{3,4}^2 A_{1,2}^6$  we supplement the reduced system by additional conditions coming from the equality of multiplicities under fusion orbits. With these extra equations we can show that  $A_{3,4}^2 A_{1,2}^6$  is not a possible  $V_1$ -structure in the same manner as above.  $\square$

We remark that entry 62 in Table 1 of [S] should read  $E_{8,2} B_{8,1}$ .

## 7 Lattice vertex algebras

In this section we describe some results on lattice vertex algebras and their twisted modules.

Let  $L$  be an even lattice and  $\varepsilon : L \times L \rightarrow \{\pm 1\}$  a 2-cocycle satisfying  $\varepsilon(\alpha, \alpha) = (-1)^{(\alpha, \alpha)/2}$  and  $\varepsilon(\alpha, \beta)\varepsilon(\beta, \alpha) = (-1)^{(\alpha, \beta)}$ . Then  $\varepsilon(\alpha, 0) = \varepsilon(0, \alpha) = 1$ , i.e.  $\varepsilon$  is normalised. The twisted group algebra  $\mathbb{C}[L]_\varepsilon$  is the algebra with basis  $\{e^\alpha \mid \alpha \in L\}$  and products  $e^\alpha e^\beta = \varepsilon(\alpha, \beta) e^{\alpha+\beta}$ .

Let  $g \in \text{O}(L)$ . Then a map  $\phi_g$  on  $\mathbb{C}[L]_\varepsilon$  of the form  $\phi_g(e^\alpha) = \eta(\alpha) e^{g(\alpha)}$  for some function  $\eta : L \rightarrow \{\pm 1\}$  is an automorphism of  $\mathbb{C}[L]_\varepsilon$  if and only if  $\eta$  is a quadratic form satisfying  $\eta(\alpha + \beta)\eta(\alpha)\eta(\beta) = \varepsilon(\alpha, \beta)\varepsilon(g(\alpha), g(\beta))$ . These automorphisms form a group which is a non-split extension of  $\text{O}(L)$  by  $\text{Hom}(L, \mathbb{Z}_2)$  (cf. [B2] and [MP], Chapter 3).

Let  $g \in \text{O}(L)$ . Then for  $\alpha \in L^{g^k}$  we have

$$\eta(\alpha + g(\alpha) + \dots + g^{k-1}(\alpha)) = \eta(\alpha)\eta(g(\alpha)) \dots \eta(g^{k-1}(\alpha))$$

if  $k$  is odd and

$$\eta(\alpha + g(\alpha) + \dots + g^{k-1}(\alpha)) = (-1)^{(\alpha, g^{k/2}(\alpha))} \eta(\alpha)\eta(g(\alpha)) \dots \eta(g^{k-1}(\alpha))$$

if  $k$  is even.

There is a lift  $\phi_g$  of  $g$  such that  $\eta = 1$  on the fixed-point sublattice  $L^g$  of  $L$ . We call such a lift a standard lift of  $g$ .

**Proposition 7.1**

Suppose  $g$  has odd order  $n$  and let  $\phi_g$  be a standard lift of  $g$ . Then

$$\phi_g^k(e^\alpha) = e^\alpha$$

for all  $\alpha \in L^{g^k}$ . In particular  $\phi_g$  has order  $n$ .

*Proof:* First suppose that  $k$  is odd. Then

$$\begin{aligned} \phi_g^k(e^\alpha) &= \eta(\alpha)\eta(g(\alpha)) \dots \eta(g^{k-1}(\alpha))e^\alpha \\ &= \eta(\alpha + g(\alpha) + \dots + g^{k-1}(\alpha))e^\alpha \\ &= e^\alpha \end{aligned}$$

because  $\alpha + g(\alpha) + \dots + g^{k-1}(\alpha)$  is in  $L^g$  and  $\eta = 1$  on  $L^g$ . Now suppose that  $k$  is even. Then  $k + n$  is odd and  $e^\alpha = \phi_g^{k+n}(e^\alpha) = \phi_g^k(\phi_g^n(e^\alpha)) = \phi_g^k(e^\alpha)$  because  $\phi_g$  has order  $n$ .  $\square$

In the same way one shows

**Proposition 7.2**

Suppose  $g$  has even order  $n$  and let  $\phi_g$  be a standard lift of  $g$ . Then for all  $\alpha \in L^{g^k}$

$$\phi_g^k(e^\alpha) = e^\alpha$$

if  $k$  is odd and

$$\phi_g^k(e^\alpha) = (-1)^{(\alpha, g^{k/2}(\alpha))} e^\alpha$$

if  $k$  is even. In particular  $\phi_g$  has order  $n$  if  $(\alpha, g^{n/2}(\alpha))$  is even for all  $\alpha \in L$  and  $2n$  otherwise.

Let  $h = L \otimes \mathbb{C}$  and  $\hat{h}$  be the Heisenberg algebra corresponding to  $h$ . Then the extension of  $O(L)$  described above acts naturally on the vertex algebra  $V = S(\hat{h}^-) \otimes \mathbb{C}[L]_\varepsilon$  corresponding to  $L$ .

We assume now that  $L$  is unimodular. Let  $L^{g^\perp}$  be the orthogonal complement of  $L^g$  in  $L$ . Then the orthogonal projection  $\pi : h \rightarrow h$  of  $h$  on  $L^g \otimes \mathbb{C}$  sends  $L$  to the dual  $L^{g'}$  of  $L^g$ . Let  $\eta$  be a quadratic form on  $L$  as above. Then  $\eta$  is a homomorphism on  $L^g$  and there is an element  $s_\eta \in \pi(h)$  such that  $\eta(\alpha) = e((s_\eta, \alpha))$  for all  $\alpha \in L^g$ . The element  $s_\eta$  is unique up to  $L^{g'} = \pi(L)$ . A minor variation on the arguments in [BK] and [DL2] which deal with the case of standard lifts only leads to

**Theorem 7.3**

Let  $L$  be an even unimodular lattice and  $g$  an automorphism of  $L$  of order  $n$ . Let  $\eta : L \rightarrow \{\pm 1\}$  be a quadratic form as above and  $\phi_g$  the associated lift of  $g$  to an automorphism of  $\mathbb{C}[L]_\varepsilon$ . Then the unique irreducible  $\phi_g$ -twisted  $V_L$ -module is isomorphic as a vector space to

$$S(\hat{h}_g^-) \otimes e^{s_\eta} \mathbb{C}[\pi(L)] \otimes X$$

where  $X$  is a complex vector space of dimension  $d$  with  $d^2 = |L^{g^\perp}/(1-g)L|$  and  $S(\hat{h}_g^-)$  the twisted Fock space. Under this identification the  $L_0$ -eigenvalue of the vector

$$(a_1(-n_1) \dots a_l(-n_l)) \otimes e^{s_\eta + \pi(\alpha)} \otimes x$$

is equal to  $\sum n_i + (s_\eta + \pi(\alpha))^2/2 + \rho$  where

$$\rho = \frac{1}{4n^2} \sum_{j=1}^{n-1} j(n-j) \dim(h_j).$$

Here  $h_j$  denotes the  $e(j/n)$ -eigenspace of  $g$  in  $h$ .

We need the description of twisted modules for non-standard lifts because sometimes  $\phi_g^m$  is not a standard lift of  $g^m$  if  $\phi_g$  is a standard lift of  $g$ . For example in the case of the order 4 orbifold in the next section  $\phi_g^2$ , where  $\phi_g$  is a standard lift of  $g$ , is not a standard lift of  $g^2$ .

## 8 Construction of some new holomorphic vertex operator algebras

In this section we construct 5 new holomorphic vertex operator algebras of central charge 24 as orbifolds of lattice vertex algebras.

We proceed as follows. Let  $N(\Phi)$  be a Niemeier lattice with root system  $\Phi$  and  $g$  an automorphism of  $N(\Phi)$  of order  $n$ . We take a standard lift of  $g$  to the vertex algebra  $V$  of  $N(\Phi)$  which we also denote by  $g$ . Then the twisted modules  $V(g^j)$  have positive conformal weights for  $j \neq 0$  and we can compute the type of  $g$  using Theorem 7.3. We assume that  $g$  has type 0. Then the twisted traces  $T(\mathbf{1}, i, j, \tau)$  can be determined from the known characters  $T(\mathbf{1}, 0, j, \tau)$  and the explicit formulas for the  $\mathcal{S}$ - and  $\mathcal{T}$ -matrix given in the remark after Theorem 5.15. We have

$$\chi_{W^{(i,0)}}(\tau) = T_{W^{(i,0)}}(\mathbf{1}, \tau) = \frac{1}{n} \sum_{j \in \mathbb{Z}_n} T(\mathbf{1}, i, j, \tau)$$

so that the dimension of  $V_1^{\text{orb}(G)}$  is the constant coefficient of

$$\chi_{V^{\text{orb}(G)}}(\tau) = \frac{1}{n} \sum_{i,j \in \mathbb{Z}_n} T(\mathbf{1}, i, j, \tau).$$

Then Theorem 6.4 gives the possible Lie algebra structures of  $V_1^{\text{orb}(G)}$ . We can further restrict the structure of  $V_1^{\text{orb}(G)}$  by an argument which is due to Montague [M]. There is an automorphism  $k$  of order  $n$  acting on  $V^{\text{orb}(G)}$  such that the corresponding orbifold  $(V^{\text{orb}(G)})^{\text{orb}(K)}$  gives back  $V$ . The fixed-point subalgebra is  $(V^{\text{orb}(G)})^K = V^G$  so that  $V_1^G$  is the fixed-point subalgebra of an automorphism of  $V_1^{\text{orb}(G)}$  whose order divides  $n$ . Kac classified the finite order

automorphisms of finite-dimensional simple Lie algebras and their fixed-point subalgebras (cf. Theorem 8.6 and Proposition 8.6 in [K]). This reduces the possible affine structures of  $V_1^{\text{orb}(G)}$ . In two cases we need additional arguments to completely fix this structure.

### The affine structure $A_{4,5}^2$ as a $\mathbb{Z}_5$ -orbifold

The lattice  $A_4$  can be written as

$$A_4 = \{ (x_1, \dots, x_5) \in \mathbb{Z}^5 \mid x_1 + \dots + x_5 = 0 \} \subset \mathbb{R}^5.$$

The dual lattice is given by

$$A'_4 = \bigcup_{i=0}^4 ([i] + A_4)$$

where  $[i] = [\frac{i}{5}, \dots, \frac{i}{5}, -\frac{j}{5}, \dots, -\frac{j}{5}]$  with  $j$  components equal to  $\frac{i}{5}$  and  $i + j = 5$ . The lattice  $L = A_4^6$  has an automorphism  $g$  of order 5 obtained by acting with a permutation of order 5 on the coordinates of the first  $A_4$ -component and a permutation of order 5 of the remaining  $A_4$ -components. The characteristic polynomial of  $g$  is  $(x-1)^{-1}(x^5-1)^5$ , i.e.  $g$  has cycle shape  $1^{-1}5^5$ . Let  $H$  be the isotropic subgroup of  $L'/L$  generated by the glue vectors  $[1(01441)]$  (see [CS], Chapter 16). Then the lattice

$$N(A_4^6) = \bigcup_{\gamma \in H} (\gamma + L)$$

is a Niemeier lattice with root system  $A_4^6$  and  $g$  defines an automorphism of  $N(A_4^6)$ . The fixed-point sublattice of  $g$  in  $N(A_4^6)$  is isomorphic to  $A'_4(5)$ .

Let  $V$  be the vertex operator algebra corresponding to  $N(A_4^6)$ . We take a standard lift of  $g$  which we also denote by  $g$ . Then  $g$  has order 5 and type 0 and

$$T(\mathbf{1}, 0, i, \tau) = \text{tr}_V g^i q^{L_0-1} = \frac{\theta_{N(A_4^6)g^i}(\tau)}{\eta_{g^i}(\tau)}$$

where  $\eta_{g^i}$  denotes the eta-product corresponding to  $g^i$ . For  $i \neq 0$  we have

$$T(\mathbf{1}, 0, i, \tau) = \frac{\theta_{A'_4(5)}(\tau)}{\eta(\tau)^{-1}\eta(5\tau)^5} = 5 + \frac{\eta(\tau)^6}{\eta(5\tau)^6}$$

so that

$$T(\mathbf{1}, i, 0, \tau) = T(\mathbf{1}, 0, i, -1/\tau) = 5 + 5^3 \frac{\eta(\tau)^6}{\eta(\tau/5)^6}.$$

Write

$$5 + 5^3 \frac{\eta(\tau)^6}{\eta(\tau/5)^6} = g_0(\tau) + g_1(\tau) + \dots + g_4(\tau)$$

with  $g_j(T\tau) = e(j/5)g_j(\tau)$ . Then

$$\begin{aligned}\chi_{W^{(i,0)}}(\tau) &= \frac{1}{5} \sum_{j \in \mathbb{Z}_5} T(\mathbf{1}, i, j, \tau) = \frac{1}{5} \sum_{j \in \mathbb{Z}_5} T(\mathbf{1}, (i, 0)T^{i^{-1}j}, \tau) \\ &= \frac{1}{5} \sum_{j \in \mathbb{Z}_5} T(\mathbf{1}, i, 0, T^{i^{-1}j}\tau) = \frac{1}{5} \sum_{j \in \mathbb{Z}_5} T(\mathbf{1}, i, 0, \tau + j) \\ &= g_0(\tau) = 5 + 39375q + 4298750q^2 + 172860000q^3 + \dots\end{aligned}$$

for  $i \neq 0$ . Since  $G = \langle g \rangle$  acts fixed-point freely on the set of roots of  $N(A_4^6)$  we have

$$\dim W_1^{(0,0)} = \dim V_1^G = \frac{120}{5} + 4 = 28.$$

It is not difficult to see that  $W_1^{(0,0)} = V_1^G$  is isomorphic as a Lie algebra to  $A_4\mathbb{C}^4$  with  $\mathbb{C}^4$  coming from the 4 orbits of  $G$  on the 20 roots of the first  $A_4$ -component. Let

$$V^{\text{orb}(G)} = \bigoplus_{i \in \mathbb{Z}_5} W^{(i,0)}$$

be the orbifold of  $V$  corresponding to  $g$ . Then  $V_1^{\text{orb}(G)}$  has dimension  $48 = 28 + 4 \cdot 5$  and by Theorem 6.4 is isomorphic as a Lie algebra to  $A_1^{16}$ ,  $A_2^6$ ,  $A_1A_3^3$ ,  $A_4^2$ ,  $A_1A_5B_2$ ,  $A_1D_5$  or  $A_6$ . The simple components in this list which admit an automorphism of order dividing 5 whose fixed-point subalgebra contains  $A_4$  are  $A_4$ ,  $A_5$ ,  $D_5$  and  $A_6$ . The corresponding fixed-point subalgebras are  $A_4$ ,  $A_4\mathbb{C}$ ,  $A_4\mathbb{C}$  and  $A_4\mathbb{C}^2$ . Hence  $V_1^{\text{orb}(G)} = A_4^2$  or  $A_1A_5B_2$ . The conformal weight of the spaces  $W^{(i,0)}$ ,  $i \neq 0$  is 1. By Lemma 2.2.2 in [SS] this implies that  $A_4 \subset V_1^G$  is not only a subalgebra but an ideal in  $V_1^{\text{orb}(G)}$ . Hence  $V^{\text{orb}(G)}$  has the affine structure  $A_{4,5}^2$ .

### The affine structure $C_{4,10}$ as a $\mathbb{Z}_{10}$ -orbifold

Again we consider the Niemeier lattice  $N(A_4^6)$  with root system  $A_4^6$ . Let  $g$  be the product of the automorphism of the previous example with  $-1$ . Then  $g$  has order 10 and cycle shape  $1^{12}2^{-1}5^{-5}10^5$ . The inner product  $(\alpha, g^k(\alpha))$  is even for all  $\alpha \in N(A_4^6)^{g^{2k}}$ . We take a standard lift of  $g$  to the vertex algebra  $V$  of  $N(A_4^6)$  which we also denote by  $g$ . Then  $g$  has order 10 and type 0. The space  $V_1^{\text{orb}(G)}$  has dimension 36 and is isomorphic as a Lie algebra to  $A_1^{12}$ ,  $A_2D_4$  or  $C_4$ . The Lie algebra structure of  $V_1^G$  is  $B_2\mathbb{C}^2$ . This excludes the first possibility. Recall that the automorphism  $k$  on  $V^{\text{orb}(G)}$  acts as  $kv = e(i/10)v$  for  $v \in W^{(i,0)}$ . Hence the eigenspace of  $k^2$  on  $V^{\text{orb}(G)}$  is

$$(V^{\text{orb}(G)})^{K^2} = W^{(0,0)} \oplus W^{(5,0)}.$$

Since  $\dim(W_1^{(5,0)}) = 0$  this implies that  $V_1^G = W_1^{(0,0)}$  is the fixed-point subalgebra of  $V_1^{\text{orb}(G)}$  of an automorphism whose order divides 5 and not just 10. It follows  $V_1^{\text{orb}(G)} = C_4$ .

### The affine structure $A_{1,1}C_{5,3}G_{2,2}$ as a $\mathbb{Z}_6$ -orbifold

Recall that the lattice  $E_6$  has 3 conjugacy classes of automorphisms of order 3 of cycle shape  $1^{-3}3^3$ ,  $3^2$  and  $1^33^1$ . The lattice  $L = E_6^4$  has an automorphism of order 3 acting by a fixed-point free automorphism of order 3 on the first  $E_6$ -component and a permutation of order 3 of the remaining  $E_6$ -components. Let  $g$  be the product of this automorphism with  $-1$ . Then  $g$  has cycle shape  $1^32^{-3}3^{-9}6^9$ . Let  $H$  be the isotropic subgroup of  $L'/L$  generated by the glue vectors  $[1(012)]$  (see [CS], Chapter 16). Then the lattice

$$N(E_6^4) = \bigcup_{\gamma \in H} (\gamma + L)$$

is a Niemeier lattice with root system  $E_6^6$  and  $g$  defines an automorphism of  $N(E_6^4)$  satisfying  $(-1)^{(\alpha, g^k(\alpha))} = 1$  for all  $\alpha \in N(E_6^4)^{g^{2k}}$ .

Let  $V$  be the vertex operator algebra corresponding to  $N(E_6^4)$ . We take a standard lift of  $g$  to  $V$  which we also denote by  $g$ . Then  $g$  has order 6 and type 0. Let  $V_1^{\text{orb}(G)}$  be the corresponding orbifold. Then  $\dim(V_1^{\text{orb}(G)}) = 72$  so that  $V_1^{\text{orb}(G)}$  is isomorphic as a Lie algebra to  $A_1^{24}$ ,  $A_1^4A_3^4$ ,  $A_1^3A_5D_4$ ,  $A_1^2C_3D_5$ ,  $A_1^3A_7$ ,  $A_1C_5G_2$  or  $A_1^2D_6$ . The Lie algebra structure of  $V_1^G$  is  $A_1A_2C_4\mathbb{C}^1$ . The only simple components in the list for  $V_1^{\text{orb}(G)}$  which admit an automorphism of order dividing 6 whose fixed-point subalgebra contains  $C_4$  are  $A_7$  and  $D_5$ . This implies that  $V_1^{\text{orb}(G)} = A_1^3A_7$  or  $A_1C_5G_2$ . In the first case  $C_4$  is the full fixed-point subalgebra of  $A_7$  so that  $A_2$  would have to be a subalgebra of  $A_1$  which is impossible. Hence  $V_1^{\text{orb}(G)} = A_1C_5G_2$ .

### The affine structure $A_{2,1}B_{2,1}E_{6,4}$ as a $\mathbb{Z}_4$ -orbifold

The lattice  $D_6 = \{(x_1, \dots, x_6) \in \mathbb{Z}^6 \mid x_1 + \dots + x_6 = 0 \pmod{2}\} \subset \mathbb{R}^6$  has an automorphism of order 4 defined by  $(x_1, x_2, x_3, x_4, x_5, x_6) \mapsto (x_1, x_2, x_4, x_3, -x_6, x_5)$ . Composing this map with the automorphism  $(x, y) \mapsto (-y, x)$  on  $A_9^2$  we obtain an automorphism  $g$  of the Niemeier lattice  $N(A_9^2D_6)$  (see [CS], Chapter 16) of cycle shape  $1^22^{-9}4^{10}$ . Here  $(-1)^{(\alpha, g(\alpha))} = -1$  for some elements  $\alpha \in N(A_9^2D_6)^{g^2}$  while  $(-1)^{(\alpha, g^2(\alpha))} = 1$  for all  $\alpha \in N(A_9^2D_6)$ . We take a standard lift of  $g$  to the vertex algebra  $V$  of  $N(A_9^2D_6)$  which we also denote by  $g$ . Then  $g$  has order 4 and type 0. The dimension of  $V_1^{\text{orb}(G)}$  is 96 so that  $V_1^{\text{orb}(G)} = A_2^{12}$ ,  $B_2^4D_4^2$ ,  $A_2^2A_5^2B_2$ ,  $A_2^2A_8$  or  $A_2B_2E_6$ . The Lie algebra structure of  $V_1^G$  is  $A_2B_2D_5\mathbb{C}^1$ . Out of the list of simple components of  $V_1^{\text{orb}(G)}$  only  $E_6$  admits an automorphism of order dividing 4 whose fixed-point subalgebra contains  $D_5$ . This implies  $V_1^{\text{orb}(G)} = A_2B_2E_6$ .

### The affine structure $A_{2,6}D_{4,12}$ as a $\mathbb{Z}_6$ -orbifold

The lattice  $A_2$  has a fixed-point free automorphism of order 3 which we denote by  $f$ . We define an automorphism of  $A_2^{12} = \{(x_1, \dots, x_{12}) \mid x_i \in A_2\}$  of order 6



by composing the maps

$$\begin{aligned}
(x_2, x_3, x_7, x_9, x_{10}, x_{12}) &\mapsto (x_3, x_{12}, x_9, x_{10}, x_2, x_7) \\
(x_6, x_8, x_{11}) &\mapsto (-x_8, -x_{11}, -x_6) \\
(x_1, x_5) &\mapsto (-f(x_5), -f(x_1)) \\
x_4 &\mapsto -f(x_4)
\end{aligned}$$

This automorphism has cycle shape  $1^1 2^{-2} 3^{-3} 6^6$  and defines an automorphism of the Niemeier lattice  $N(A_2^{12})$  (see [CS], Chapter 16) because it preserves the glue group. Furthermore  $(-1)^{(\alpha, g^k(\alpha))} = 1$  for all  $\alpha \in N(A_2^{12})^{g^{2k}}$ . We take a standard lift of this automorphism to the vertex algebra  $V$  of  $N(A_2^{12})$  which we denote by  $g$ . Then  $g$  has order 6 and type 0. The Lie algebra structure of  $V_1^G$  is  $A_1 A_2 \mathbb{C}^3$ . For the dimension of  $V_1^{\text{orb}(G)}$  we find  $\dim(V_1^{\text{orb}(G)}) = 36$  so that  $V_1^{\text{orb}(G)} = A_1^{12}, A_2 D_4$  or  $C_4$ . This implies  $V_1^{\text{orb}(G)} = A_2 D_4$ .

## Lattice orbifolds

We summarise our results in the following theorem.

### Theorem 8.1

*There exist holomorphic vertex operator algebras of central charge  $c = 24$  with the following 5 affine structures.*

Aff. structure	No. in [S]	Niemeier lat.	Aut. order
$A_{2,1} B_{2,1} E_{6,4}$	28	$A_9^2 D_6$	4
$A_{4,5}^2$	9	$A_4^6$	5
$A_{2,6} D_{4,12}$	3	$A_2^{12}$	6
$A_{1,1} C_{5,3} G_{2,2}$	21	$E_6^4$	6
$C_{4,10}$	4	$A_4^6$	10

In [LS3] Lam and Shimakura give constructions of holomorphic vertex operator algebras with 2 new affine structures conditionally upon the existence of those with  $A_{4,5}^2$  and  $A_1 C_{5,3} G_{2,2}$  (and 3 others unconditionally). Hence

### Corollary 8.2

*There exist holomorphic vertex operator algebras of central charge  $c = 24$  with the affine structures  $A_1 D_{6,5}$  and  $A_{5,6} C_{2,3} A_{1,2}$ .*

Finally we review the list of known holomorphic  $c = 24$  vertex operator algebras.

- 24 vertex operator algebras associated with the Niemeier lattices (cf. [B1], [FLM])

- 15  $\mathbb{Z}_2$ -orbifolds using standard lifts of the  $(-1)$ -involution (see [FLM] for the moonshine module  $V^\natural$ , [DGM] for the others)
- 17 framed vertex operator algebras (cf. [L], [LS1] and [LS2])
- 3 lattice  $\mathbb{Z}_3$ -orbifolds (cf. [M1], [ISS] and [SS])
- 5 orbifolds by inner automorphisms (see [LS3]), 2 of them conditional upon the next item)
- 5 lattice  $\mathbb{Z}_n$ -orbifolds with  $n = 4, 5, 6, 10$  (present work)

Thus at least 69 of the 71 affine structures of Schellekens' list (Theorem 6.4) are realised by holomorphic vertex operator algebras. The two remaining affine structures are  $A_{6,7}$  and  $A_{2,2}F_{4,6}$ .

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